Diffusive Motion and Recurrence on an Idealized Galton Board

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We study a mechanical model known as a Galton board—a particle rolling on a tilted plane under gravitation and bouncing off a periodic array of rigid pegs. Incidentally, this model is identical to a periodic Lorentz gas where an electron is driven by a uniform electric field. Previous heuristic and experimental studies have suggested that the particle's speed $v(t)$ should grow as $t^{1/3}$ and its coordinate $x(t)$ as $t^{2/3}$. We find exact limit distributions for the rescaled velocity $t^{-1/3}v(t)$ and position $t^{-2/3}x(t)$. In addition, we determine that the particle's motion is recurrent; i.e., the particle comes back to the top of the board with a probability of one.

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Introduction.—The Galton board introduced in [1] is one of the simplest mechanical devices where nonstationary transport occurs. It consists of a vertical (or inclined) board with interleaved rows of pegs. A ball thrown into the Galton board moves under gravitation and bounces off the pegs on its way down.

Galton board has been extensively studied in various asymptotic regimes; see [2–4] and references therein. In this Letter we discuss an idealized infinite Galton board; our ball is a point particle of unit mass moving according to equations $\dot{\mathbf{q}} = \mathbf{v}$ and $\dot{\mathbf{v}} = \mathbf{g} = \text{const}$ and colliding elastically with immobile convex obstacles of infinite mass (scatterers), which are positioned periodically on the board. We assume that every straight line intersects some obstacles, so that there are no collision-free corridors; see Fig. 1 (this is a standard "finite horizon" condition that guarantees a diffusive behavior of the ball). We neglect friction and the spin of the ball.

This model is identical to a 2D periodic Lorentz gas [5–8], which illustrates the transport of electrons in metals in a spatially homogeneous electric field. Without an external field (i.e., when $\mathbf{g} = 0$), the periodic Lorentz gas reduces to a billiard system on its fundamental domain (a torus minus scatterers). This is known as a dispersing (or Sinai) billiard [9]; it has a stationary Liouville measure and strong statistical properties: the position $\mathbf{q}(t)$ of the Lorentz particle at time t evolves as a 2D Brownian motion [10], in particular, $\mathbf{q}(t)/\sqrt{t} \to \mathcal{N}(0, \mathbf{D})$, where $\mathbf{D}$ is a diffusion matrix determined by the geometry of scatterers.

Under a constant external field in the x direction, i.e., $\mathbf{g} = (g, 0)$, the moving particle is allowed to accelerate indefinitely; thus, the system does not have a stationary state, but it conserves the total energy

$$E = \tfrac{1}{2}[v(t)]^2 - gx(t) = \text{const}, \tag{1}$$

where $v(t)$ is the particle's speed and $x(t)$ its displacement in the direction of the field. Thus the farther the particle travels, the faster it moves. On the other hand, higher speed leads to a stronger scattering effect, thus increasing the chances that the particle bounces back and hence temporarily decelerates (this is similar to Fermi, or diffusive shock acceleration [11,12]).

The backscattering effect slows down the particle's drift in the x direction so much that its average displacement $\langle x(t)\rangle$ at time t will only grow as t^a with some $a < 1$. It was estimated [13–16] by heuristic and approximative arguments, as well as computer simulation, that the displacement of the particle typically grows as $t^{2/3}$. Because of the conservation of energy, its speed then grows as $t^{1/3}$.

We derive these conjectures from the equations of motion and recent results [7,8], and we precisely describe the limit distributions for the rescaled velocity $t^{-1/3}v(t)$ and the rescaled position $t^{-2/3}x(t)$. We also show that this mechanical model, after a proper rescaling of space and

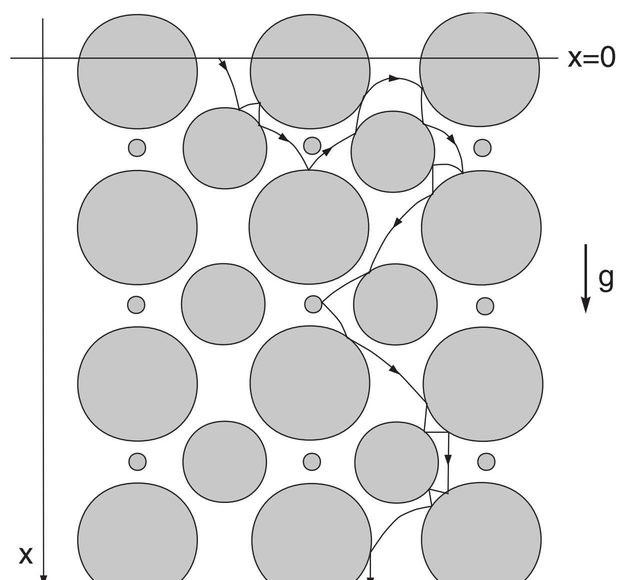

FIG. 1. A ball's trajectory on a Galton board.

time, is governed by a certain set of stochastic differential equations. This provides a complete solution to the classical Galton problem. In addition, we find, quite surprisingly, that the particle's motion is recurrent; that is, with probability one, the particle must slow down and return to the top of the board.

Our approach is quite general. It relies only on chaoticity of the dynamics (for large kinetic energies) and the Einstein relation for linear response. Therefore it should be useful in other problems such as particles in plasma [17,18], dynamics of electrons in antidot superlattice [19], balls falling onto a moving plate [20], to mention just a few. In this Letter we apply our method to a relatively simple model, so that the computations are very explicit and transparent. Still we only present the core of our argument here, full proofs with mathematical details will be published elsewhere [21].

Velocity distribution.—We consider two types of Galton boards: an ''open top'' board, where the ball bouncing back to the top escapes, and a ''closed lid'' board where the ball hitting the closed lid reflects back down.

We assume that the ball starts on the line $x = 0$ with its initial velocity $\mathbf{v}(0)$ pointing in the (general) x direction, and its initial speed $v(0)$ must be high enough (then, in the closed board, it will stay high at all times, due to the conservation of energy). The initial state of the ball is chosen randomly via a smooth probability distribution.

We obtain two major facts: (A) In the ''open top'' board the ball escapes with probability one. (B) In the ''closed lid'' board, the limit distribution of $ct^{-1/3}v(t)$, for some $c > 0$, has probability density

$$\frac{3z}{\Gamma(2/3)}\exp[-z^3], \qquad z \geq 0. \tag{2}$$

Accordingly, the limit distribution of $2gc^2t^{-2/3}x(t)$ has density

$$\frac{3}{2\Gamma(2/3)}\exp[-z^{3/2}], \qquad z \geq 0. \tag{3}$$

In addition, $x(t)$ returns to zero infinitely many times with probability one.

The last statement means that the Galton particle evolves in a recurrent manner—its *excursions* into the depth of the Galton board alternate with *retreats* to the starting line $x = 0$. As time goes on, the particle makes longer and longer excursions that extend farther and farther into the board (because the average coordinate $\langle x(t)\rangle$ must grow as $t^{2/3}$), but every excursion is followed by a retreat of the particle back onto the starting line.

To derive our results, we approximate the dynamics of the Galton particle (whose kinetic energy $K = v^2/2$ may grow indefinitely) with an isokinetic particle moving at fixed speed. To this end we rescale time $t \to t/\sqrt{\varepsilon}$, where $\varepsilon \sim K^{-1}$, which brings our system to the form where the kinetic energy $\tilde{K} = \varepsilon K$ is of order one, but the force is weak $\mathbf{g} \to \varepsilon\mathbf{g}$. In other words, we get a so-called slow-fast system, with a slow variable $\tilde{K}$ and a pair of fast variables $X = (q, \boldsymbol{\omega})$, where $\boldsymbol{\omega} = \mathbf{v}/v$ denotes the particle direction. In these variables, the rescaled equations of motion read

$$\dot{q} = \sqrt{2\tilde{K}}\boldsymbol{\omega}, \qquad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2\tilde{K}}}[\mathbf{g} - \langle\mathbf{g}, \boldsymbol{\omega}\rangle\boldsymbol{\omega}] + \mathcal{O}(\varepsilon^2),$$
$$\dot{\tilde{K}} = \varepsilon\sqrt{2\tilde{K}}\langle\mathbf{g}, \boldsymbol{\omega}\rangle. \tag{4}$$

Now we approximate (4) by an isokinetic system

$$\dot{q} = \sqrt{2K}\boldsymbol{\omega}, \qquad \dot{\boldsymbol{\omega}} = \frac{\varepsilon}{\sqrt{2K}}[\mathbf{g} - \langle\mathbf{g}, \boldsymbol{\omega}\rangle\boldsymbol{\omega}], \qquad \dot{K} = 0. \tag{5}$$

The advantage of this approximation is that the dynamics on any energy surfaces can be reduced to that on the unit speed surface. Namely, the solution to (5) with initial condition $(q_0, \boldsymbol{\omega}_0, K_0)$ takes the form

$$K(t) = K_0,$$
$$(q, \boldsymbol{\omega})(t, \varepsilon, q_0, \boldsymbol{\omega}_0, K_0) = (\hat{q}, \hat{\boldsymbol{\omega}})(t\sqrt{2K_0}, \varepsilon/2K_0, q_0, \boldsymbol{\omega}_0),$$

where $(\hat{q}, \hat{\boldsymbol{\omega}})(t, \varepsilon, q_0, \boldsymbol{\omega}_0)$ denotes the solution of

$$\dot{\hat{q}} = \hat{\boldsymbol{\omega}}, \qquad \dot{\hat{\boldsymbol{\omega}}} = \varepsilon[\mathbf{g} - \langle\mathbf{g}, \hat{\boldsymbol{\omega}}\rangle\hat{\boldsymbol{\omega}}] \tag{6}$$

with initial condition $(q_0, \boldsymbol{\omega}_0)$. Equations (6) describe a particle in a periodic Lorentz gas under a constant external field $\varepsilon\mathbf{g}$ moving at unit speed due to a Gaussian thermostat; this model was introduced in [14] and studied in [7,8]. It is known that the dynamics (6) has a steady state μ_ε and satisfies the central limit theorem: for any observable A

$$\int_0^T A(\hat{q}(t), \hat{\boldsymbol{\omega}}(t))dt = T\mu_\varepsilon(A) + \sqrt{T}\sigma_\varepsilon(A)\mathcal{Z} + o(\sqrt{T}), \tag{7}$$

where $\mathcal{Z} = \mathcal{N}(0, 1)$ is a standard normal random variable and $\mu_\varepsilon(A)$ and $\sigma_\varepsilon(A)$ are the asymptotic drift and diffusion (standard deviation). In addition, *Ohm's Law* is derived in [7,8]:

$$\mu_\varepsilon(\hat{\boldsymbol{\omega}}) = \tfrac{1}{2}\varepsilon\mathbf{D}\mathbf{g} + o(\varepsilon), \tag{8}$$

where again $\mathbf{D} = \sigma_0^2(\hat{\boldsymbol{\omega}})$. The analysis of [7,8] relies heavily on the fact that (6) is a small perturbation of the Sinai billiard, which corresponds to $\varepsilon = 0$. In particular, the diffusion matrix depends continuously on the force strength:

$$\sigma_\varepsilon(\hat{\boldsymbol{\omega}}) = \sigma_0(\hat{\boldsymbol{\omega}}) + o(1). \tag{9}$$

Our facts (A) and (B) actually follow from a more general result: (C) Let $\bar{K} \geq 0$. Suppose the initial state $[X(0), \tilde{K}(0)]$ of our particle (in the closed Galton board) is chosen randomly via a probability distribution such that $\tilde{K}(0) = \bar{K}$, then the rescaled kinetic energy $\tilde{K}(\tau\varepsilon^{-2})$, where $0 < \tau < 1$ is a new slow time, is approximated (for small ε) by an Itô diffusion process $\mathcal{K}(\tau) \geq 0$ satisfying the stochastic differential equation (SDE)

$$d\mathcal{K} = \frac{\sigma^2}{2\sqrt{2\mathcal{K}}} d\tau + (2\mathcal{K})^{1/4}\sigma dW_\tau, \qquad \mathcal{K}(0) = \bar{K}, \tag{10}$$

where W_τ is the standard Brownian motion and $\sigma^2 = \langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle$.

Equation (10) has a notable singularity at 0, which can be eliminated by changing variable $\mathcal{Q} = \mathcal{K}^{3/2}$, after which standard facts ([22], Section IX.3) guarantee the existence and uniqueness of $\mathcal{Q}$ and $\mathcal{K}$. Actually, $\mathcal{Q}$ is known as a square Bessel process of index $-1/3$; see [22]. For the reader's convenience, we derive (A) and (B) from (C) in the Appendix.

A crucial property of Eq. (10) is its self-similarity: it remains invariant under the transformation $t \to ct$, $\mathcal{K} \to c^{2/3}\mathcal{K}$. As a result, not only the rescaled kinetic energy $\tilde{K}$, but the original one K as well, is approximated by (10); in fact one can study the evolution of $K(t)$ for $0 < t < T$, by substituting $\varepsilon = T^{-2/3}$ in (C).

We now derive (C) from (4)–(9). Let $T = \delta\varepsilon^{-2}$ with a small $\delta > 0$; then approximations (4)–(6) give

$$\tilde{K}(T) - \tilde{K}(0) \approx \varepsilon\sqrt{2\bar{K}} \int_0^T \langle \mathbf{g}, \boldsymbol{\omega}\rangle dt \approx \varepsilon \int_0^{\hat{T}} \langle \mathbf{g}, \hat{\boldsymbol{\omega}}\rangle dt,$$

where $\hat{T} = T\sqrt{2\bar{K}}$. Using (7)–(9) we obtain

$$\tilde{K}(T) - \tilde{K}(0) \approx \frac{\langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle\delta}{2\sqrt{2\bar{K}}} + (2\bar{K})^{1/4}\sqrt{\delta}\langle \mathbf{g}, \sigma_0(\hat{\boldsymbol{\omega}})Z^{(2)}\rangle,$$

where $Z^{(2)}$ denotes a normal 2-vector; also observe that $\langle \mathbf{g}, \sigma_0(\hat{\boldsymbol{\omega}})Z^{(2)}\rangle = \langle \mathbf{g}, \mathbf{D}\mathbf{g}\rangle^{1/2} Z$. Likewise, if we divide a longer time interval $(0, \tau\varepsilon^{-2})$ into segments of size $\delta\varepsilon^{-2}$, we obtain

$$\tilde{K}_{j+1} - \tilde{K}_j \approx \frac{\sigma^2\delta}{2\sqrt{2\tilde{K}_j}} + (2\tilde{K}_j)^{1/4}\sigma\sqrt{\delta}Z_j, \tag{11}$$

where $\tilde{K}_j = \tilde{K}(j\delta\varepsilon^{-2})$ and Z_j are independent. Now (11) is just a discrete approximation to (10).

Coordinate distribution.—We also determine the limit distribution for the y coordinate of the Galton particle. Let $\mathbf{h}$ be a unit vector in the y direction. For simplicity, assume that the periodic array of pegs is symmetric about the x axis, so that the Lorentz gas diffusion matrix $\mathbf{D}$ is diagonal, i.e., $\langle \mathbf{h}, \mathbf{D}\mathbf{g}\rangle = 0$. Let $\sigma_y^2 = \langle \mathbf{h}, \mathbf{D}\mathbf{h}\rangle$. For the rescaled system (4), we have $d\tilde{y}/dt = \varepsilon\langle \mathbf{v}, \mathbf{h}\rangle$, where $\tilde{y} = \sqrt{\varepsilon}y$. Now the same analysis as in the previous section shows that $\tilde{y}$ can be approximated by the solution of SDE

$$\begin{aligned} d\mathcal{Y}(\tau) &= (2\mathcal{K})^{1/4}\sigma_y d\tilde{W}_\tau + \frac{\langle \mathbf{h}, \mathbf{D}\mathbf{g}\rangle}{2\sqrt{2\mathcal{K}}} d\tau \\ &= (2\mathcal{K})^{1/4}\sigma_y d\tilde{W}_\tau \end{aligned} \tag{12}$$

with $\mathcal{Y}(0) = 0$; here $\tilde{W}_\tau$ stands for a standard 1D Brownian motion independent from W [thus (10) naturally decouples from (12)].

For any fixed realization of $\mathcal{K}(\tau)$, the conditional distribution of $\mathcal{Y}(\tau)$ is such that its increments are independent and normal:

$$\mathcal{Y}(\tau + \Delta) - \mathcal{Y}(\tau) = \mathcal{N}(0, \sigma_y^2\sqrt{2\mathcal{K}(\tau)}\Delta) + o(\Delta);$$

therefore $\mathcal{Y}(\tau)$ is (conditionally) a Gaussian random variable with zero mean and variance $\sigma_y^2 \int_0^\tau \sqrt{2\mathcal{K}(\zeta)}d\zeta$. Thus, $\mathcal{Y}(\tau)/(\int_0^\tau \sqrt{2\mathcal{K}(\zeta)}d\zeta)^{1/2}$ is normal $\mathcal{N}(0, \sigma_y^2)$ and independent of $\mathcal{K}(\tau)$.

As a result, $t^{-2/3}y(t)$ is a product of two independent random variables Y_1Y_2, where $Y_1 = \mathcal{N}(0, \sigma_y^2)$ and $Y_2 = (\int_0^1 \sqrt{2\mathcal{K}(\zeta)}d\zeta)^{1/2}$, and $\mathcal{K}$ is the solution of (10) starting at 0. We see that $y(t) \sim t^{2/3}$.

Last, we estimate the expected number of times the particle collides with a given scatterer. In order to hit a scatterer during a time interval $[n, n+1]$, the particle needs to be at a distance $\mathcal{O}(1)$ from it at time n, and this event has probability $p_n \sim \mathcal{O}(n^{-4/3})$, since the distributions of both the x and y coordinates have standard deviation of order $n^{2/3}$. Since $\sum p_n < \infty$, the expected number of returns to any given scatterer is finite. This indicates that the coordinate process is not recurrent.

Three dimensional model.—Our arguments should work in higher dimension; furthermore, in 3D the analogues of (10) and (12) can be solved explicitly, so the results are even easier to formulate. For simplicity we assume that the periodic array of scatterers is symmetric about each coordinate plane, so that the corresponding Lorentz gas diffusion matrix is again diagonal. Let W_1, W_2, W_3, and W_4 be some independent 1D Brownian motion processes. Then our analysis shows that (i) the velocity process is recurrent; (ii) the coordinate process is not recurrent; (iii) there are constants $c_1, c_2, c_3 > 0$, such that the rescaled coordinate vector $t^{-2/3}(c_1x(t), c_2y(t), c_3z(t))$ converges in distribution to $[(W_1^2(1) + W_2^2(1))^{2/3}, \Lambda W_3(1), \Lambda W_4(1)]$ where

$$\Lambda = \left[\int_0^1 (W_1^2(s) + W_2^2(s))^{1/3} ds\right]^{1/2}.$$

Appendix.—To derive (A) and (B) from (C) we use elements of Itô calculus [22]. An Itô diffusion process satisfies a SDE

$$dX = a(X,t)dt + b(X,t)dW_t, \qquad X(0) = X_0, \tag{13}$$

where $a(X,t)$ is the drift coefficient and $b(X,t)$ is the diffusion coefficient [the solution of (13) is a time-homogeneous Markov process with continuous paths]. If a and b do not depend on t, the Fokker-Plank equation for this process reads

$$\frac{\partial\rho}{\partial t} = \frac{1}{2}\left[\frac{\partial}{\partial x}\right]^2 (b^2\rho) - \frac{\partial}{\partial x}(a\rho). \tag{14}$$

Consider another process $Y = \lambda(X,t)$, where λ is a smooth function. The Itô formula asserts that

$$dY = \left[\lambda' a + \tfrac{1}{2}\lambda'' b^2 + \dot{\lambda} \right] dt + \lambda' b dW_t, \qquad (15)$$

where the primes stand for space derivatives and the dots for time derivatives (in particular, Y is also an Itô diffusion process).

Another useful tool is changing the time variable: introducing new time $dt = \kappa(X, t)ds$ transforms (13) into $dX = a\kappa ds + b\sqrt{\kappa} dW_s$. Now combining (10) and (15) shows that the process $\mathcal{W} = \sqrt{\mathcal{K}}$ satisfies $d\mathcal{W} = \frac{\sigma}{2^{3/4}\mathcal{W}^{1/2}} dW_\xi$, and changing the time by $d\eta = \frac{\sigma^2}{2^{3/2}\mathcal{W}} d\xi$ gives $d\mathcal{W} = dW_\eta$; i.e., $\mathcal{W}(\eta)$ is a standard Brownian motion. The latter is a recurrent process, hence so is our $\mathcal{K}$, which implies the fact (A).

Next, the process $\mathcal{R} = \xi^{-2/3}\mathcal{K}$ satisfies SDE

$$d\mathcal{R} = \left[\frac{1}{2\sqrt{2\mathcal{R}}} - \frac{2\mathcal{R}}{3} \right] \frac{d\xi}{\xi} - \frac{(2\mathcal{R})^{1/4}}{\sqrt{\xi}} dW_\xi.$$

Changing time via $d\zeta = d\xi/\xi$ gives

$$d\mathcal{R} = \left[\frac{1}{2\sqrt{2\mathcal{R}}} - \frac{2\mathcal{R}}{3} \right] d\zeta - (2\mathcal{R})^{1/4} dW_\zeta.$$

The Fokker-Plank equation for $\mathcal{R}$ reads [see (14)]

$$\frac{\partial \rho}{\partial \zeta} = \left[\frac{\partial}{\partial r} \right]^2 (\sqrt{2r}\rho) - \frac{\partial}{\partial r}\left(\left[\frac{1}{2\sqrt{2r}} - \frac{2r}{3} \right] \rho \right).$$

It is clear that any time independent integrable solution of this equation must satisfy

$$\frac{\partial}{\partial r}(\sqrt{2r}\rho) = \left[\frac{1}{2\sqrt{2r}} - \frac{2r}{3} \right] \rho;$$

thus the asymptotic density of $\mathcal{K}$ is (3). Last, (2) follows from (1).

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