Is There a "Most Perfect Fluid" Consistent with Quantum Field Theory?

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(Received 20 February 2007; published 9 July 2007)

It was recently conjectured that the ratio of the shear viscosity to entropy density η/s for any fluid always exceeds $\hbar/(4\pi k_B)$. A theoretical counterexample to this bound can be constructed from a nonrelativistic gas by increasing the number of species in the fluid while keeping the dynamics essentially independent of the species type. The question of whether the underlying structure of relativistic quantum field theory generically inhibits the realization of such a system and thereby preserves the possibility of a universal bound is considered here. Using rather conservative assumptions, it is shown here that a metastable gas of heavy mesons in a particular controlled regime of QCD provides a realization of the counterexample and is consistent with a well-defined underlying relativistic quantum field theory. Thus, quantum field theory appears to impose no lower bound on η/s, at least for metastable fluids.

DOI: 10.1103/PhysRevLett.99.021602 PACS numbers: 11.10.Wx, 05.60.−k, 12.38.Mh

A universal bound for the ratio of the shear viscosity η to entropy density s in *any* fluid has recently been conjectured by Kovtun, Son, and Starinets (KSS) [1]:

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B}, \tag{1}$$

where the k_B and $\hbar$ are Boltzmann's constant and Planck's constant, respectively. This is the strong form of the KSS conjecture. Since the limit of zero viscosity is the "perfect" fluid, the bound from the strong KSS conjecture implies that "the most perfect" fluid is one with $\eta = \hbar s/(4\pi k_B)$. For simplicity, in the remainder of this Letter, units will be chosen using standard theory conventions with $\hbar$ and k_B set to unity. The KSS conjecture in its strong form represents an extremely important advance in our understanding of many-body physics, if correct. Indeed, it has been invoked in discussing physics as diverse as ultracold gases of trapped atoms [2] and the quark-gluon plasma [3].

A fluid is a substance that continually deforms (i.e., flows) under an applied shear stress regardless of the magnitude of the applied stress. Usually, this definition is taken to hold regardless of whether the state of the system is stable or metastable (in the sense of being very long-lived on all relevant time scales). Consider a liquid high explosive—which obviously is not absolutely stable. Such a substance is usually regarded as unambiguously being a fluid. In principle, η and s can be measured for a liquid high explosive with extremely high accuracy despite its lack of absolute stability (although great care is advised in making the measurements). There may be a fundamental uncertainty in values of η and s arising from a natural time scale for the lifetime of the metastable state τ_{met}. However, one expects that any fundamental uncertainty to be of relative order $\tau_{\mathrm{fl}}/\tau_{\mathrm{met}}$, where τ_{fl} is the natural time scale of the fluid dynamics (e.g., the collision time for a dilute system). Since this ratio is many orders of magnitude less than unity for typical metastable fluids, the entropy density and viscosity are essentially well defined. It is natural to suppose that if the KSS bound is a generic feature of fluids that it will also hold for metastable fluids (up to the very small intrinsic uncertainty in the value of η and s due to metastability), although it is logically possible that the bound holds only for stable fluids.

A principal motivation for the conjecture is based on the transport properties on gauge theories with gravity duals computed [4] using the famed anti-de Sitter–conformal field theory correspondence [5]. A truly remarkable result has emerged [1]. For systems at nonzero temperature, at large N_c, and infinitely strong 't Hooft coupling $g^2 N_c$, all known theories with a gravity dual have the same ratio of the shear viscosity (η) to the entropy density (s): $\frac{\eta}{s} = \frac{1}{4\pi}$. This suggests a general feature of classical gravity which should hold for all large N_c theories with gravity duals; in Refs. [6,7] it was shown to hold in a very wide class of theories. It is natural to assume that η/s increases as one departs from the limit of infinite strong 't Hooft coupling since weaker couplings should increase η. An explicit calculation confirmed this behavior for a specific model [8]. Thus, a class of large N_c gauge theories with gravity duals appears to have a bound for η/s. The KSS conjecture is that the bound could be much more general. Moreover, the conjectured bound does not involve c (when units are restored), and it is not unnatural that the bound, if true, applies equally to nonrelativistic systems [1]. A heuristic argument for the bound in nonrelativistic systems can be found in Ref. [9].

The bound is respected by all real fluids considered [1,7]. Typically, for a fluid at fixed pressure and low temperatures, η/s decreases with increasing temperature until a minimum is reached and then increases. The minima all appear to be well above the putative bound, providing an empirical basis for the conjecture.

However, in a second publication [7] KSS briefly note a serious threat to this universal conjecture based on rather elementary theoretical considerations indicating a class of counterexample. The issue raised there is critical to the discussion, and an expanded variant of this argument is given below.

Consider a nonrelativistic quantum many-body system in a regime for which the computation of η/s is analytically tractable. The system consists of a gas comprised of a number (N_s) of distinct but equivalent species of spin-zero bosons of degenerate mass. The dynamics is specified by a Hamiltonian containing kinetic energy terms plus two-body interactions which are independent of the species type: For all species a and b, $V_{ab}(r) = V(r/R)$, where R is the range of the potential. The gas is in local thermal equilibrium and has an equal density of each species: $n_a = n/N_s$ for any species a. The system is in the regime

$$R^{-2}, a^{-2} \gg mT \gg n^{2/3}, \tag{2}$$

where a is the scattering length. This simultaneous low temperature, low density regime can be enforced by a common scaling behavior with a parameter ξ:

$$n = \frac{n_0}{\xi^4}, \qquad T = \frac{T_0}{\xi^2}. \tag{3}$$

Sufficiently large ξ ensures the regime of Eq. (2). At large ξ, n is low enough for the entropy to be given by the classical ideal gas (with small corrections). Moreover, the many-body dynamics is dominated by binary collisions and hence is in the regime of validity of the Boltzmann equation [10]. Although this regime has slowly moving particles yielding long quantum wavelengths for the thermal particles, one expects the dynamics between collisions to be essentially classical (as required for the Boltzmann equation). The key point is that, while the thermal wavelength scales as ξ, the interparticle spacing scales as $\xi^{4/3}$; thus, at large ξ, the quantum wavelengths are much shorter than the relevant distance scale, and the system behaves classically between collisions. The low temperature aspect implies that the two-body scattering amplitude of a typical pair is predominantly s-wave and (to good approximation) equal to the scattering at zero momentum. Thus, the scattering term in the Boltzmann equation is based on isotropic scattering independent of energy. This is formally equivalent to the Boltzmann equation for *classical* hard sphere scattering, a case for which the shear viscosity is readily computed: $\eta = C_{\rm hs}\sqrt{mT}/d^2$, where d is the diameter and $C_{\rm hs} \approx 0.179$ is a coefficient calculable numerically [10]. The ratio η/s is universal in this regime and is given by

$$\frac{\eta}{s} = \frac{C_{\rm hs}\xi^3\sqrt{mT_0}}{a^2 n_0[\log(\frac{(mT_0)^{3/2}}{n_0}) + \frac{5}{2} + \log(\xi) + \log(N_s)]}. \tag{4}$$

Based on the general reasoning above, one expects that corrections to Eq. (4) are suppressed by various powers of $1/\xi$ and should be irrelevant at sufficiently large ξ. Ideally, this should be checked by systematic calculations of the leading corrections to verify explicitly that they are power suppressed. However, it is not clear that this is tractable at present, and the analysis here will be based on the general theoretical arguments given above. The validity of Eq. (4) at large ξ should not depend on the number of species being small. Accordingly, it is legitimate to consider an exponentially large number of species: Take N_s to be

$$N_s = \exp(\xi^4) \tag{5}$$

(rounded to the nearest whole number). As the temperature and density of the system are decreased, the number of species simultaneously increases greatly. In the combined regime of Eqs. (3) and (5), η/s is given as

$$\frac{\eta}{s} = \frac{1}{\xi}\frac{C_{\rm hs}\sqrt{mT_0}}{a^2 n_0} \tag{6}$$

up to power law corrections. Clearly, η/s can be made arbitrarily small by increasing ξ and, in particular, can be made smaller than $1/(4\pi)$. At large ξ, the mixing entropy associated with the many species overwhelms other effects yielding a small value of η/s.

For purely repulsive potentials, the argument appears to hold generally. As a practical matter, it would be very difficult to realize this example in realistic circumstances—the number of equivalent species must be the exponent of a large number. However, it does show as a matter of principle that nothing in ordinary many-body quantum mechanics requires that the conjectured strong KSS bound be respected.

Subtleties arise if the potential has attractive regions. The second derivative of the free energy density is given by $d^2f/dn^2 = \xi^2(T_0/n_0) - 4\pi|a|/m$ (up to power law corrections in ξ^{-1}). It is clearly positive for large ξ, indicating that the fluid is *locally* stable against density fluctuations. However, it may be possible for the system to globally lower its energy either by forming macroscopic regions of higher density or by forming two- or higher-body bound states. Regardless of whether the system is absolutely stable, the time scale for the decay of the system is very long at large ξ. The fastest possible mechanism for the system to qualitatively alter its state is if two-body bound states exist. To conserve energy and momentum, two particles can bind only if three or more particles interact. This implies that in this situation $\tau_{\rm fl}/\tau_{\rm met} \sim nR_0^3 \sim \xi^{-4}$, and the system becomes a well-defined fluid at large ξ. For other mechanisms, the lifetime of the metastable system is parametrically larger. Thus, regardless of whether the interaction is attractive or repulsive, a fluid violating the KSS bound can be constructed (although for an attractive interaction it may only be metastable.)

It is useful to introduce another variant of this example which will be of use later. Suppose that, in addition to N_s, n, and T, the mass and the strength of the interaction potential also have a nontrivial scaling with ξ:

$$\begin{gathered} m = m_0\exp(\xi^4), \qquad R = R_0, \\ V_{ab}(r) = V_o(r/R_0)\exp(-\xi^4), \qquad T = \frac{T_0\exp(-\xi^4)}{\xi^2}, \\ n = \frac{n_0}{\xi^4}, \qquad N_s = \exp(\xi^4). \end{gathered} \tag{7}$$

In this scaling, the particles become heavy at the same rate as the number of species grows, while their interaction potentials become weak at the same rate. The two-body scattering depends only on the product mV and is unaltered by this scaling; thus, the cross section remains independent of ξ. Note that T in Eq. (7) drops off exponentially faster than the analogous scaling in Eq. (3). However, the relevant combination is the thermal momentum $\sqrt{mT}$, which scales as ξ^{-1} in both scaling regimes. Thus, for this scaling regime η/s is still given by Eq. (6) with m replaced by m_0; corrections are still down by powers of ξ^{-1}, and the KSS bound is still violated at large ξ.

One way to accommodate counterexamples of this sort is to discard the strong form of the conjecture in favor of a much weaker version—any relativistic quantum field theoretic system at zero chemical potential has the ratio of $\eta/s \geq (4\pi)^{-1}$—as proposed by KSS in Ref. [7]. The weaker form of the conjecture remains of great theoretical importance, if true. However, its practical implications are severely limited; apart from relativistic heavy ion collisions where the chemical potential may be low enough to neglect, the vast majority of fluid problems of physical interest are outside of its domain. Moreover, the weak form of the conjecture seems less plausible if the strong form is discarded. A principal justification for the conjecture is empirical data on ordinary fluids; this provides no support for the weak version. In addition, it has recently been shown [11] that systems with gravity duals at infinite 't Hooft coupling and large N_c with nonzero chemical potentials also have $\eta/s = 1/(4\pi)$ just as with zero chemical potential. To the extent that the zero chemical potential case motivates the weak form of the conjecture, one might expect a similar level of justification to hold at nonzero chemical potential.

However, the very fact that the counterexamples are so impractical raises an obvious question: Could there be some deep physical reason of principle preventing them from being realized? This could preserve the strong form of the conjecture. Since this physics must be beyond the structure of nonrelativistic quantum mechanics, it is natural to ask if the structure of the underlying relativistic quantum field theory could do this. The issue is whether it is possible for nonrelativistic systems inconsistent with the strong KSS conjecture to arise from any consistent underlying quantum field theory, i.e., whether there exists a consistent ultraviolet completion for the nonrelativistic system [12].

A priori, it is not implausible that the structure of quantum field theory might rule out these counterexamples and preserve the strong form of the conjecture. Consider a simple example: a low temperature, low density pion gas in a generalization of QCD with N_f degenerate flavors—half of which we denote to be"uplike" and half "downlike." The net number density for uplike quarks and downlike antiquarks can be set to n, and the temperature can be chosen to be well below m_π, yielding a nonrelativistic pion gas with $N_f^2/4$ species of pion present. However, it is not possible to take N_F large (and thus evade the KSS bound) while keeping the theory well defined. The one-loop β function for QCD is $\beta(g) = -(g^3/16\pi^2)[(11N_c/3) - (2N_f/3)]$; asymptotic freedom is lost at large N_f, and the theory presumably becomes ill defined in the ultraviolet [13]. One might evade this by setting N_c large simultaneously with N_f with the ratio N_f/N_c held fixed. However, $\pi - \pi$ scattering cross section scales as N_c^{-2}, implying that $\eta \sim N_c^2$. Thus, either the theory is ill defined in the ultraviolet or η/s grows with increasing species number. If all consistent field theories behave in an analogous way, then the strong KSS conjecture would remain viable. However, this behavior is not generic.

Consider a nonrelativistic gas of heavy mesons (i.e., mesons with the quantum numbers of a heavy quark and a light antiquark) in a very special regime of generalized QCD with weak and electromagnetic interactions absent. The regime involves a theory with N_c colors, N_h degenerate flavors of heavy quark (of mass m_h), and one flavor of light quark (of mass m_l). The total density of light antiquarks to be n and the density of each heavy flavor is fixed at n/N_h (for a total heavy meson density of n). The parameters of the theory are taken to scale according to

$$N_c = e^{\xi^4}, \quad N_h = e^{\xi^4}, \quad m_h = m_{h0}e^{\xi^4}, \quad m_l \sim m_{l0},$$
$$\Lambda_{\mathrm{QCD}} = \Lambda_{\mathrm{QCD0}}, \quad n = n_0\xi^{-4}, \quad T = T_0\frac{e^{-\xi^4}}{\xi^2}. \tag{8}$$

The scaling relations in Eqs. (8) are designed to create a nonrelativistic gas of heavy mesons in a regime equivalent to that of Eq. (7). The problem of diminishing cross section in the large N_c, large N_f pion gas is evaded here by having the meson mass grow to compensate for the decreased interaction strength yielding a constant cross section.

There are several aspects of this system which require comment. First, note that the scaling rules give $N_f = N_h + 1 = N_c + 1$. Asymptotic freedom is maintained, and the theory remains well defined at large ξ (assuming, of course, that QCD itself is, in fact, well defined).

The next key point is that the system behaves like a nonrelativistic gas whose constituents are the lightest pseudoscalar mesons containing a heavy quark. The pseudoscalar containing a quark of flavor a is denoted as H_a. Recall that, for heavy quarks, the vector meson $H_a^\star$ is nearly degenerate [14] with the pseudoscalar H_a: $(M_H^\star - M_H) \sim \Lambda_{\mathrm{QCD}}^2/m_H \sim e^{-\xi^4}(\Lambda_{\mathrm{QCD}}^2/m_{h0})$. Combining this with the scaling of the temperature, one sees that $(M_H^\star - M_H)/T \sim \xi^2(\Lambda_{\mathrm{QCD}^2}/m_{h0})$ so that, at sufficiently large ξ, excitations of the $H^\star$ are thermally suppressed; the probability of a heavy meson being in the $H^\star$ state is exponentially suppressed by a factor of $e^{-\xi^2}$. Other excited states are suppressed by *much* larger factors (i.e., by exponentials of exponentials of ξ). Similarly, the density of light mesons in the system is thermally suppressed by a very large factor at large ξ. Thus, both the viscosity and the entropy of the system are dominated by the H mesons. Moreover, the H mesons are clearly extremely nonrelativistic: The characteristic thermal velocity $v_T = \sqrt{T/M_H}$ scales as $e^{-\xi^4}/\xi^2$.

Only the dynamics of H mesons are relevant at large ξ; all other degrees of freedom can be integrated out, yielding an effective theory. The appropriate effective theory is nonrelativistic quantum mechanics with an effective potential between the H mesons; the particle number is essentially fixed and the motion is nonrelativistic. From standard large N_c analysis, it is easy to see that the strength of the effective potential scales like $1/N_c$; the range of the effective potential is fixed from the *light* degrees of freedom (light quark and gluons) since virtual heavy quarks may be integrated out for large ξ. Thus, in the regime of Eq. (8), the system behaves like a nonrelativistic gas of pseudoscalar heavy mesons; their properties scale analogously to those in Eqs. (7). From the logic leading to Eq. (6), one expects that $\frac{\eta}{s} = \frac{1}{\xi}(C_{\rm hs}\sqrt{m_{h0}T_0}/a^2 n_0)$, where a is the meson-meson scattering length (corrections are power law suppressed in $1/\xi$). The KSS bound is violated at large ξ since η/s becomes arbitrary small.

Given this counterexample, it appears that the strong form of the KSS conjecture cannot be supported: A fluid based on an underlying consistent relativistic field theory can have an arbitrarily small value of η/s. Of course, the preceding argument does not rule out the strong form of the KSS conjecture in any rigorous mathematical sense. The calculation of η/s in this regime was based on numerous approximations. However, these approximations appear to be well motivated physically, and the result should be exact in the limit of large ξ; accordingly, it seems highly unlikely that there exists a fundamental bound on the "most perfect" fluid which is consistent with an underlying quantum field theory.

There is an important caveat to this conclusion. In the heavy meson system, there are attractive interactions between, and the fluid can be shown to be metastable rather than stable (with $\tau_{\rm fl}/\tau_{\rm met}$ going to infinity at large ξ). The logical possibility exists that the strong KSS conjecture holds but only for stable fluids. *A priori,* it is unclear how plausible it is for the KSS bound to hold for stable fluids but not to hold (even approximately) for metastable fluids no matter how long-lived. Moreover, if the strong KSS bound holds only for absolutely stable systems, it cannot be applied directly to ordinary fluids such as water which are *not* stable under the standard model. In principle, the free energy of water can be lowered via nuclear reactions (e.g., two hydrogen nuclei combining from a deuteron). One cannot argue that the standard model dynamics are irrelevant (on the grounds that the dynamics are essentially purely electromagnetic in nature); the logic supporting the strong KSS conjecture depended on the existence of an underlying quantum field theory. Thus, the direct application [15] of the strong KSS bound to ordinary fluids is legitimate only if the bound applies to metastable fluids; the previous counterexample implies that it does not.

The author has greatly benefited from numerous insightful comments by D. T. Son. The support of the U.S. Department of Energy is gratefully acknowledged.

[1] P. Kovtun, D. T. Son, and A. O. Starinets, J. High Energy Phys. 10 (2003) 064.

[2] T. Schaefer, arXiv:cond-mat/0701251.

[3] A. Adare *et al.* (PHENIX Collaboration), arXiv:nucl-ex/0611018; T. Hirano and M. Gyulassy, Nucl. Phys. **A769**, 71 (2006); T. Hirano, U. Heinz, D. Kharzeev, R. Lacey, and Y. Nara, Phys. Lett. B **636**, 299 (2006); J. L. Nagle, Eur. Phys. J. C **49**, 275 (2007).

[4] G. Policastro, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **87**, 081601 (2001); J. High Energy Phys. 12 (2002) 054; C. P. Herzog, J. High Energy Phys. 12 (2002) 026.

[5] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[6] A. Buchel and J. T. Liu, Phys. Rev. Lett. **93**, 090602 (2004).

[7] P. Kovtun, D. T. Son, and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).

[8] A. Buchel, J. T. Liu, and A. O. Starinets, Nucl. Phys. **B707**, 56 (2005).

[9] B. A. Gelman, E. V. Shuryak, and I. Zahed Phys. Rev. A **72**, 043601 (2005).

[10] The Boltzmann equation and its application to the hard sphere gas is standard kinetic theory. See, for example, E. M. Lifshitz and L. P. Pitaevskii, *Physical Kinetics* (Pergamon Press, Oxford, 1981) or R. L. Liboff, *Kinetic Theory: Classical, Quantum and Relativistic Descriptions* (Springer-Verlag, New York, 2003), 3rd ed. The calculation of $C_{\rm hs}$ can be done in the Chapman-Enskog expansion. At leading order in the Chapman-Enskog expansion $C_{\rm hs} = 5/(16\sqrt{pi})$, higher order terms increase it by about 1.5%.

[11] K. Maeda, M. Natsuume, and T. Okamura, Phys. Rev. D **73**, 066013 (2006); O. Saremi, J. High Energy Phys. 10 (2006) 083; D. T. Son and A. O. Starinets, J. High Energy Phys. 03 (2006) 052; P. Benincasa, A. Buchel, and R. Naryshkin, Phys. Lett. B **645**, 309 (2007); J. Mas, J. High Energy Phys. 03 (2006) 016.

[12] The possibility that nonrelativistic systems inconsistent with the strong KSS bound may have no consistent ultraviolet completion was suggested by Dam Son (private communication).

[13] J. W. Chen and E. Nakano, arXiv:hep-ph/0604138.

[14] N. Isgur and M. B. Wise Phys. Lett. B **232**, 113 (1989).

[15] D. T. Son (private communication) has pointed out that it may be possible to apply the bound *indirectly* for ordinary fluids. The idea is to construct a quantum field theory with electrons and fundamental fields for nuclei and with electromagnetic interactions which look like those in the standard model and therefore yield essentially the same fluid properties. However, it is unclear whether such a theory can be found which is both well defined and yields absolutely stable fluids.