

Black Hole Thermodynamics from Euclidean Horizon Constraints

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To explain black hole thermodynamics in quantum gravity, one must introduce constraints to ensure that a black hole is actually present. I show that for a large class of black holes, such “horizon constraints” allow the use of conformal field theory techniques to compute the density of states, reproducing the Bekenstein-Hawking entropy in a nearly model-independent manner. One standard string theory approach to black hole entropy arises as a special case, lending support to the claim that the mechanism may be “universal.” I argue that the relevant degrees of freedom are Goldstone-boson-like excitations arising from the weak breaking of symmetry by the constraints.

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Thirty years ago, Bekenstein and Hawking showed us that black holes are thermodynamic systems, with characteristic temperatures and entropies [1]. This poses a problem: in Wheeler’s words, black holes have “no hair,” no classical degrees of freedom that could account for such thermal behavior. The most likely resolution is that the relevant degrees of freedom are purely quantum mechanical, in which case black hole statistical mechanics might offer important insights into quantum gravity.

Until recently, little was known about such quantum degrees of freedom. Today, we suffer an embarrassment of riches. The microscopic states of a black hole can be counted in at least three ways, not obviously equivalent, in string theory [2–4], in two rather different approaches to loop quantum gravity [5,6], in induced gravity [7], and perhaps in causal set models [8]. Each of these approaches has limitations, but each seems to work within its domain of applicability. The new puzzle is that despite counting very different degrees of freedom, they all give the same answer. More generally, in the absence of any classical degrees of freedom to which one could apply the correspondence principle, it is not obvious why *any* counting of microscopic states should reproduce Hawking’s original semiclassical results.

One possible answer, first suggested in [9], is that a classical symmetry near the black hole horizon might control the number of degrees of freedom, independent of the details of quantum gravity. To an observer outside a black hole, the near-horizon region is effectively two dimensional and conformally invariant [10]: transverse excitations and dimensionful quantities are redshifted away relative to the degrees of freedom in the r - t plane. Such a conformal description is powerful enough to determine the flux and spectrum of Hawking radiation [11]. But as Cardy has shown [12], the density of states, and thus the entropy, of a two-dimensional conformal field theory are completely fixed by a small number of parameters, independent of the particulars of the theory.

In this Letter, I compute these parameters for a very large class of black holes and obtain the usual Bekenstein-

Hawking entropy. The key step is the imposition of “horizon constraints” to ensure the presence of a black hole. I show that one standard string theory approach occurs as a special case, offering support for the claim of “universality.” Finally, I argue that conformal methods suggest a physical picture of the relevant degrees of freedom, as Goldstone-boson-like excitations arising from the breaking of symmetry by the conformal anomaly.

Euclidean dilaton gravity.—If the relevant near-horizon degrees of freedom are two dimensional, a two-dimensional effective field theory description should exist. For the case of spherical symmetry, the Kaluza-Klein reduction to two dimensions is well understood. It is less well known that a similar reduction exists even without symmetries [13], albeit at the expense of introducing an enormous gauge group.

Given such an effective theory, we must next ensure that a black hole is actually present, by imposing boundary conditions or constraints that imply the existence of a horizon. The “horizon as boundary” approach of [9] has had some success, but involves transformations that diverge at the horizon [14]. A direct imposition of horizon constraints was attempted in [15], but ambiguities remain, traceable to the lightlike nature of the horizon.

To avoid these difficulties, I will follow the common path of analytically continuing to “Euclidean” gravity. This shrinks the horizon from a light cone $t^2 - r_*^2 = 0$ to a point $t^2 + r_*^2 = 0$, and makes time periodic. As in string theory, I will consider “radial quantization,” evolution outward in a radial direction. After a field redefinition, the two-dimensional dilaton gravity action is then [16]

$$I = \frac{1}{2} \int d^2x \sqrt{g} \left[\varphi R + V[\varphi] - \frac{1}{2} W[\varphi] h_{IJ} F_{ab}^I F^{Jab} \right], \quad (1)$$

where F_{ab}^I is a non-Abelian gauge field strength, $V[\varphi]$ and $W[\varphi]$ are arbitrary potentials, and my units are $8\pi G = 1$. For dimensionally reduced gravity, the dilaton field φ is the transverse area; in general, the entropy of a black hole is proportional to its value φ_H at the horizon.

Introducing an ADM-like [17] decomposition of the metric

$$ds^2 = N^2 f^2 dr^2 + f^2 (dt + \alpha dr)^2 \quad (2)$$

and transforming to (radial) Hamiltonian form, one finds that the Hamiltonian is, as usual, a sum of constraints,

$$\begin{aligned} \mathcal{H}_{\parallel} &= \dot{\varphi} \pi_{\varphi} - f \dot{\pi}_f = 0, \\ \mathcal{H}_{\perp} &= f \pi_f \pi_{\varphi} + f \left(\frac{\dot{\varphi}}{f} \right)' - \frac{1}{2} f^2 \hat{V} = 0, \\ \mathcal{H}_I &= \dot{\pi}_I - c_{IK}^J A^K \pi_J = 0, \end{aligned} \quad (3)$$

with $A^I = A_I^I$ and $\hat{V} = V + h^{IJ} \pi_I \pi_J / W$. By the \mathcal{H}_I constraints, $h^{IJ} \pi_I \pi_J$ is a constant, so the net effect of the gauge field is simply to shift V . The constraints \mathcal{H}_{\parallel} and \mathcal{H}_{\perp} can be combined into two complex generators,

$$\begin{aligned} L[\xi] &= \frac{1}{2} \int dt \xi (\mathcal{H}_{\parallel} + i \mathcal{H}_{\perp}), \\ \bar{L}[\xi] &= \frac{1}{2} \int dt \xi (\mathcal{H}_{\parallel} - i \mathcal{H}_{\perp}). \end{aligned} \quad (4)$$

These generate holomorphic and antiholomorphic diffeomorphisms, the fundamental symmetries of a conformal field theory, and satisfy the Virasoro algebra [18]

$$\begin{aligned} \{L[\xi], L[\eta]\} &= L[\xi \dot{\eta} - \eta \dot{\xi}] + \frac{c}{48\pi} \int dt (\dot{\xi} \ddot{\eta} - \dot{\eta} \ddot{\xi}), \\ \{L[\xi], \bar{L}[\eta]\} &= 0, \\ \{\bar{L}[\xi], \bar{L}[\eta]\} &= \bar{L}[\xi \dot{\eta} - \eta \dot{\xi}] + \frac{\bar{c}}{48\pi} \int dt (\dot{\xi} \ddot{\eta} - \dot{\eta} \ddot{\xi}), \end{aligned} \quad (5)$$

with central charges $c = \bar{c} = 0$.

Horizon constraints.—We next demand that a point $r = r_H$ be a black hole horizon. The necessary conditions were worked out by Teitelboim [19]: we require $s = \bar{s} = 0$ at r_H , where

$$s = f \pi_f - i \dot{\varphi}. \quad (6)$$

Unfortunately, our radial foliation breaks down at r_H , so we must be slightly indirect, requiring a “stretched horizon” at $r_H + \epsilon$ and then take the limit $\epsilon \rightarrow 0$.

To find the appropriate stretched horizon conditions, note that a black hole is characterized by two quantities, the expansion ϑ and the surface gravity $\hat{\kappa}$. The expansion, the fractional rate of change of the transverse area, is simply $\vartheta = s/\varphi$. The surface gravity is trickier: it depends on the normalization of the null normal to the horizon, which in turn depends on asymptotic behavior [20]. We can initially define $\kappa = \pi_{\varphi} - i\dot{f}/f$. Then under a rescaling of the null normal, or equivalently a conformal transformation $f^2 \rightarrow e^{2\omega} f^2$ (with the conformal factor ω to be determined later), we obtain

$$\hat{\kappa} = \kappa + f^2 \frac{d\omega}{d\varphi} = \pi_{\varphi} - i\dot{f}/f + f^2 \frac{d\omega}{d\varphi}. \quad (7)$$

As our horizon constraints, we now choose

$$K = s - a(\hat{\kappa} - \hat{\kappa}_H) = 0, \quad \bar{K} = \bar{s} - a(\bar{\kappa} - \bar{\kappa}_H) = 0, \quad (8)$$

where the constant a will be determined below. The $\hat{\kappa}$ dependence is fixed by requiring the existence of a Hamiltonian for conformal transformations at the horizon [21], and ensures invariance under constant rescalings of the horizon normal [15]. Our constraints define a stretched horizon around a black hole with surface gravity $\hat{\kappa}_H$, and reduce to Teitelboim’s conditions as $\hat{\kappa} \rightarrow \hat{\kappa}_H$.

Horizon algebra and state counting.—We now make the crucial observation that the horizon constraints (8) are not preserved by the symmetries (4). We can cure this with a trick introduced by Bergmann and Komar [22]: we add “zero,” in the form of K and \bar{K} , to the Virasoro generators to produce new generators that commute with K and \bar{K} . Specifically, if Δ_{ij} is the inverse of $\{K_i, K_j\}$, then any modified observable

$$O^* = O - \sum_{i,j} \int dudv \{O, K_i(u)\} \Delta_{ij}(u, v) K_j(v) \quad (9)$$

will have vanishing brackets with the K_i .

The Poisson brackets of such modified observables are equivalent to Dirac brackets with K and \bar{K} treated as second class constraints. A direct calculation yields

$$\begin{aligned} \{L^*[\xi], L^*[\eta]\} &= L^*[\xi \dot{\eta} - \eta \dot{\xi}] - \frac{ia}{8} \int dt (\dot{\xi} \ddot{\eta} - \dot{\eta} \ddot{\xi}) \\ &\quad + \frac{i}{8} \int dt A (\xi \dot{\eta} - \eta \dot{\xi}), \\ \{L^*[\xi], \bar{L}^*[\eta]\} &= \int dt [B_1 (\xi \dot{\eta} - \eta \dot{\xi}) + B_2 \xi \eta], \\ \{\bar{L}^*[\xi], \bar{L}^*[\eta]\} &= \bar{L}^*[\xi \dot{\eta} - \eta \dot{\xi}] + \frac{ia}{8} \int dt (\dot{\xi} \ddot{\eta} - \dot{\eta} \ddot{\xi}) \\ &\quad - \frac{i}{8} \int dt \bar{A} (\xi \dot{\eta} - \eta \dot{\xi}), \end{aligned} \quad (10)$$

with

$$\begin{aligned} A &= a \hat{\kappa}_H^2 + f^2 \left(\hat{V} - \frac{a}{2} \frac{d\hat{V}}{d\varphi} - 2a \bar{\kappa}_H \frac{d\omega}{d\varphi} \right) + \mathcal{O}(\varphi - \varphi_H)^2, \\ B_1, B_2 &= \mathcal{O}(\varphi - \varphi_H)^2. \end{aligned} \quad (11)$$

If we now demand that the $\mathcal{O}(\varphi - \varphi_H)$ term in the anomaly A vanish, the conformal factor ω is fixed:

$$\omega = -\frac{\hat{V}_H}{4\hat{\kappa}_H} \left[\ln(\varphi \hat{V}) - \left(1 + \frac{2\varphi_H}{a} \right) \ln \varphi \right] + \mathcal{O}(\varphi - \varphi_H)^2. \quad (12)$$

With this choice, (10) is again a Virasoro algebra at the horizon, but with a shifted central charge and zero modes. Choosing a basis of periodic vector fields

$$\xi_n = \frac{\beta}{2\pi} e^{2\pi i n t / \beta}, \quad (13)$$

we can read off central charges and conformal weights $\Delta = L[\xi_0]$, $\bar{\Delta} = \bar{L}[\xi_0]$: with factors of $8\pi G$ restored,

$$c = \bar{c} = \frac{3a}{4G}, \quad \Delta = \bar{\Delta} = \frac{a}{32G} \left(\frac{\kappa_H \beta}{2\pi} \right)^2. \quad (14)$$

Cardy's formula [12] then determines the entropy:

$$S = 2\pi \sqrt{\frac{c\Delta}{6}} + 2\pi \sqrt{\frac{\bar{c}\bar{\Delta}}{6}} = \frac{\pi|a|}{4G} \left(\frac{\kappa_H \beta}{2\pi} \right). \quad (15)$$

We must still fix the constants a and β . We begin by finding the classical solutions. Defining a new function $\hat{j}[\varphi]$ by the condition $\hat{V}[\varphi] = d\hat{j}/d\varphi$, it is straightforward to show that the general static solution is

$$f^2 = s = \hat{j} - \hat{j}_H, \quad \varphi' = N(\hat{j} - \hat{j}_H), \quad \kappa = \hat{V}/2, \quad (16)$$

where N is arbitrary, \hat{j}_H is an integration constant, and a horizon occurs at $\varphi_H = \hat{j}^{-1}(\hat{j}_H)$. The metric (2) has a conical singularity at the horizon unless t has a period

$$\beta = 2\pi/\hat{\kappa}_H, \quad (17)$$

giving the usual Hawking temperature. With exceptions I will describe in the next section, we should thus take this to be the period appearing in (13) and (15).

As noted earlier, the surface gravity $\hat{\kappa}$ still has an ambiguity, which we must fix by looking at asymptotic behavior. It was observed in [23] that the metric can be put into asymptotically Schwarzschild form by a conformal transformation $e^{2\omega} = 1/\hat{j}$. We cannot use this here, since we know \hat{j} only up to an additive constant determined by a particular solution; we do not want our definition of $\hat{\kappa}$ to depend on which solution we are looking at. But for any solution, $f^2 = \hat{V}\varphi(1 - \varphi_H/\varphi) + \mathcal{O}(1 - \varphi_H/\varphi)^2$ near the horizon. We can therefore take $e^{2\omega} = 1/\hat{V}\varphi$. This precisely matches the condition (12) for anomaly cancellation, and further fixes $a = -2\varphi_H$.

The entropy (15) is thus $S = 2\pi\varphi_H/4G$. This is *almost* the standard Bekenstein-Hawking entropy; it differs by a factor of 2π . The same factor appears in [24], and has a direct physical explanation: black hole entropy counts horizon degrees of freedom at a fixed time, while we have computed the entropy at the horizon for *all* times, effectively integrating over a circle of circumference 2π .

Up to this 2π factor, the central charge (14) agrees with the horizon as boundary results of [9], and the conformal weights match a two-dimensional nonchiral version of the Komar integral of [25]. The relationship to loop quantization is less clear, but the loop horizon states in [5] trace back to an $SL(2, \mathbf{R})$ Chern-Simons theory with coupling constant $k = iA/8\pi\gamma G$. Such a Chern-Simons theory induces a Liouville theory at boundaries with central charge

$c = 6k$ [26]; for the natural value $\gamma = i$ of the Immirzi parameter, this again matches our results.

Our central charge does not, however, match that of the Bañados-Teitelboim-Zanelli (BTZ) black hole. Since the BTZ entropy calculation is a basic element of one string theory approach to black hole entropy, let us examine this case more closely.

BTZ and stringy black holes.—The BTZ black hole [26] is a rotating black hole in $(2+1)$ -dimensional gravity with cosmological constant $\Lambda = -1/\ell^2$. It has an event horizon at r_+ , an inner horizon at r_- , and surface gravity $\kappa_{\text{BTZ}} = (r_+^2 - r_-^2)/\ell^2 r_+$. It is asymptotically anti-de Sitter, with a boundary at infinity that is geometrically a flat cylinder; the asymptotic symmetry group is described by a Virasoro algebra with central charge and conformal weight

$$c = \bar{c} = \frac{3\ell}{2G}, \quad \Delta = \frac{(r_+ + r_-)^2}{16G\ell}, \quad \bar{\Delta} = \frac{(r_+ - r_-)^2}{16G\ell}. \quad (18)$$

With these parameters, the Cardy formula (15) gives the standard Bekenstein-Hawking entropy of $2\pi r_+/4G$.

This result depends on symmetries at infinity, and thus differs from the approach here. But $(2+1)$ -dimensional gravity has no local degrees of freedom, so we should be able to translate the BTZ results to the horizon. To do so, note first that the vectors generating the symmetries of the BTZ black hole differ from (13); instead,

$$\xi_n^{\text{BTZ}} \sim e^{in(t-\ell\phi)/\ell}, \quad \bar{\xi}_n^{\text{BTZ}} \sim e^{in(t+\ell\phi)/\ell}, \quad (19)$$

where ϕ is an angular coordinate. For the nonrotating case, we can change the period from $2\pi/\kappa_{\text{BTZ}} = 2\pi\ell^2/r_+$ [as in (13)] to $2\pi\ell$ [as in (19)] by restricting to modes Nn , with $N = \ell/r_+$. But it is known that such a restriction results in a new Virasoro algebra, with central charge and conformal weight $\tilde{c} = cN$, $\tilde{\Delta} = \Delta/N$ [27]. For our N , the new central charges $\tilde{c} = \bar{\tilde{c}} = (3a/4G)(\ell/r_+)$ then agree with the BTZ values (18), up to the factor of 2π discussed earlier.

The shifted conformal weights are now $\tilde{\Delta} = \bar{\tilde{\Delta}} = 2\pi(r_+^2/16G\ell)(\kappa\beta/2\pi)$. These match the BTZ values when $r_- = 0$, but differ for spinning black holes. To understand this difference, note that the angular coordinate ϕ in (19) is that of an observer at infinity. For dimensional reduction, we should instead use a corotating coordinate at the horizon, $\phi' = \phi - (r_-/r_+)\ell t$. This alters the time dependence of the modes (19), shifting their periods to $\beta_{\pm} = (1 \pm r_-/r_+)\beta$. These new periods give the “left” and “right” Hawking temperatures that appear in string theory, and, inserted into (14), reproduce the BTZ weights (18), again up to a factor of 2π .

The conformal computation of the BTZ black hole entropy can thus be viewed as a special case of our horizon constraint method, suitably translated to spatial infinity. By itself, this might not be of great significance. But a standard approach to entropy in string theory [3] uses the BTZ-like near-horizon structure of near-extremal black holes, com-

bined with various duality transformations, to reduce the problem to that of the BTZ black hole. The methods of this Letter thus encompass an important string theory approach to black hole entropy.

What are we counting?—A key advantage of the horizon constraint approach is that while it lets us count states, it does not require any detailed knowledge of the states being counted. Nevertheless, we can interpret these results as suggesting an interesting effective description of the states responsible for black hole entropy. In conventional approaches to quantum gravity, physical states are required to be invariant under diffeomorphisms; that is, in our context,

$$L[\xi]|\text{phys}\rangle = \bar{L}[\xi]|\text{phys}\rangle = 0. \quad (20)$$

If the Virasoro algebra (5) has a central charge, however, such conditions are incompatible with the Poisson brackets. We know how to weaken (20): we can require, for example, that only positive frequency modes annihilate physical states [18]. But then new states—for instance, the descendant states $L_{-n}|0\rangle$ —become physical.

This phenomenon is reminiscent of the Goldstone mechanism [28]: the conformal anomaly breaks diffeomorphism invariance, and “would-be pure gauge” degrees of freedom become physical. For asymptotically anti-de Sitter spacetimes in three and five dimensions, an explicit description of the corresponding degrees of freedom at infinity is possible [29]; one might hope for something similar at a horizon.

Perhaps the most important open question is whether we can also incorporate Hawking radiation and black hole evaporation. As noted above, one can compute the Hawking radiation spectrum from methods based on conformal anomalies of matter fields [11]. In [30], it was shown that a scalar field could be coupled to the conformal boundary degrees of freedom of the BTZ black hole to obtain Hawking radiation; if a similar coupling were possible here, it would represent major progress.

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