Leading Pollicott-Ruelle Resonances and Transport in Area-Preserving Maps

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The leading Pollicott-Ruelle resonance is calculated analytically for a general class of two-dimensional area-preserving maps. Its wave number dependence determines the normal transport coefficients. In particular, a general exact formula for the diffusion coefficient D is derived without any high stochasticity approximation, and a new effect emerges: The angular evolution can induce fast or slow modes of diffusion even in the high stochasticity regime. The behavior of D is examined for three particular cases: (i) the standard map, (ii) a sawtooth map, and (iii) a Harper map as an example of a map with a nonlinear rotation number. Numerical simulations support this formula.

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Diffusion is a paradigm of deterministic chaos, and its study is not new, dating back to Chirikov [1]. Its existence in Hamiltonian systems has been extensively established using a variety of approaches [2–7]. However, it was not clear that a satisfactory transport theory could be properly formulated. In order to understand deterministic diffusion, nonequilibrium statistical mechanics was suitably combined with dynamical system theory [8,9]. In this modern formulation, the stochastic properties of chaotic systems can be determined by the spectral properties of the Perron-Frobenius operator U. One of the most important properties is the exponential relaxation to the thermodynamic equilibrium, explained in great detail at the microscopic level. The relaxation rates γ_m, known as Pollicott-Ruelle (PR) resonances [10,11], are related to the poles z_m of the matrix elements of the resolvent $R(z) = (z - U)^{-1}$ as $z_m = e^{\gamma_m}$. These resonances are located inside the unit circle in the complex z plane, whereas the spectrum of U is confined to the unit circle because of unitarity [12]. Furthermore, the wave number dependence of the leading PR resonance determines the normal diffusion coefficient for spatially periodic systems [13,14]. These results are rigorous only for hyperbolic systems, though they have been confirmed in the high stochasticity approximation for some mixed systems such as the kicked rotor (standard map) [15], the kicked top [16], and the perturbed cat map [17]. The PR resonances are essential not only in classical dynamics but also in quantum dynamics. Recently, a microwave billiard experiment demonstrated a deep connection between quantum properties and classical diffusion through the spectral autocorrelation function [18].

In this Letter, the leading PR resonance will be calculated analytically for the general class of two-dimensional area-preserving maps

$$\begin{aligned} I_{n+1} &= I_n + Kf(\theta_n), \\ \theta_{n+1} &= \theta_n + c\alpha(I_{n+1}) \quad \mathrm{mod}2\pi, \end{aligned} \tag{1}$$

defined on the cylinder $-\pi \leq \theta < \pi$, $-\infty < I < \infty$. Here $f(\theta)$ is the impulse function, $\alpha(I) = \alpha(I + 2\pi r)$ is the rotation number, c and r are real parameters, and K is the stochasticity parameter. This map is commonly called the *radial twist map* [19] periodic in action (the nonperiodic case can be considered in the limit $r \to \infty$). Although considerable theoretical development in the study of diffusion has been achieved for the linear rotation number (LRN) case $c\alpha(I) \equiv I$ [2–6], many physically realistic systems are best described just by the nonlinear cases. Such maps have been extensively used in various areas of physics, especially in celestial mechanics [20], plasma and fluid physics [21], and astrophysics and accelerator devices [19,22]. However, the normal transport properties of such maps have not been studied previously [7].

The analysis of the map (1) is best carried out in Fourier space. The Fourier expansion of distribution function at the nth time, denoted by ρ_n, is given by

$$\rho_n(I, \theta) = \sum_m \int dq e^{i(m\theta + qI)} a_n(m, q). \tag{2}$$

The moments can be found from the Fourier amplitudes via $\langle I^p \rangle_n = (2\pi)^2[(i\partial_q)^p a_n(q)]_{q=0}$, where $a_n(q) \equiv a_n(0, q)$. The discrete time evolution of the probability density ρ is governed by the Perron-Frobenius operator U defined by $\rho_{n+1}(I, \theta) = U\rho_n(I, \theta)$. The matrix representation of U may be considered as the conditional probability density for the transition of the initial state (I', θ') to a final state (I, θ) in one time step, ruled by (1). The law of evolution of the Fourier coefficients will be given by

$$\begin{aligned} a_n(m, q) = \sum_{m'} \int dq' \mathcal{A}_m(r, c, q' - q) \\ \times \mathcal{J}_{m-m'}(-Kq') a_{n-1}(m', q'), \end{aligned} \tag{3}$$

where $a_0(m, q) = (2\pi)^{-2} \exp[-i(m\theta_0 + qI_0)]$. The Fourier decompositions of the $\alpha(I)$ and $f(\theta)$ functions are

$$\mathcal{A}_m(r, c, x) = \sum_l \delta(lr^{-1} - x)\mathcal{G}_l(r, mc), \tag{4}$$

$$\mathcal{G}_l(r,x) = \frac{1}{2\pi}\int d\theta \exp\{-i[x\alpha(r\theta) - l\theta]\}, \quad (5)$$

$$\mathcal{J}_m(x) = \frac{1}{2\pi}\int d\theta \exp\{-i[m\theta - xf(\theta)]\}. \quad (6)$$

If the rotation number $\alpha(I)$ is an odd function, then $\mathcal{G}_l(r,x)$ is a real function and $\mathcal{G}_{\pm|l|}(r,x) = \mathcal{G}_{|l|}(r,\pm x)$. The integral function $\mathcal{J}_m(x)$ assumes the following series expansions $\mathcal{J}_m(x) = \delta_{m,0} + \sum_{n=1}^{\infty} c_{m,n}x^n$, whose coefficients are given by

$$c_{m,n} = \frac{1}{2\pi}\frac{i^n}{n!}\int d\theta f^n(\theta)e^{-im\theta}. \quad (7)$$

Note that if $f(-\theta) = -f(\theta)$, the coefficients $c_{m,n}$ are real for all $\{m,n\}$ and $\mathcal{J}_{-m}(x) = \mathcal{J}_m(-x)$.

Let us consider the decomposition method of the resolvent $R(z)$ based on the projection operator techniques utilized in Ref. [23]. The law of evolution (3) can be written as $a_n(m,q) = U^n a_0(m,q)$, where U^n is given by the identity $\oint_C dz R(z) z^n = 2\pi i U^n$, where the spectrum of U is located inside or on the unit circle C around the origin in the complex z plane. The contour of integration is then a circle lying just outside the unit circle. We then introduce a mutually orthogonal projection operator $P = |q,0\rangle\langle q,0|$, which picks out this *relevant* state from the resolvent, and its complement $Q = 1 - P$, which projects on the *irrelevant* states. In order to calculate the diffusion coefficient $D = \lim_{n\to\infty}(2n)^{-1}\langle(I - I_0)^2\rangle_n$, we can decompose the projection of the resolvent $PR(z)$ into two parts: $PR(z) = PR(z)P + PR(z)Q$. The last part can be neglected because $a_0(m \neq 0, q) \propto e^{-im\theta_0}$, whose expected value disappears at random initial conditions on $[-\pi, \pi)$. Hence, the relevant law of evolution of the Fourier amplitudes, omitting initial angular fluctuations, assumes the following form:

$$a_n(q) = \frac{1}{2\pi i}\oint_C dz \frac{z^n}{z - \sum_{j=0}^{\infty} z^{-j}\Psi_j(q)} a_0(q), \quad (8)$$

where the memory functions $\Psi_j(q)$ obtained for the system (1) are given by

$$\Psi_0(q) = \mathcal{J}_0(-Kq), \quad (9a)$$

$$\Psi_1(q) = \sum_m \mathcal{J}_{-m}(-Kq)\mathcal{J}_m(-Kq)\mathcal{G}_0(r, mc), \quad (9b)$$

$$\Psi_{j\geq 2}(q) = \sum_{\{m\}}\sum_{\{\lambda\}^\dagger} \mathcal{J}_{-m_1}(-Kq)\mathcal{J}_{m_j}(-Kq)\mathcal{G}_{\lambda_1}(r, m_1c) \times \prod_{i=2}^{j}\mathcal{G}_{\lambda_i}(r, m_ic) \times \mathcal{J}_{m_{i-1}-m_i}\left[-K\left(q + r^{-1}\sum_{k=1}^{i-1}\lambda_k\right)\right]. \quad (9c)$$

Hereafter, the following convention will be used: Wave numbers denoted by *Roman indices* can take only *nonzero* integer values, whereas wave numbers denoted by *Greek indices* can take *all* integer values, including zero. For each fixed j, the sets of wave numbers are defined by $\{m\} = \{m_1, \ldots, m_j\}$ and $\{\lambda\}^\dagger = \{\lambda_1, \ldots, \lambda_j\}$, where the superscript denotes the restriction $\sum_{i=1}^{j}\lambda_i = 0$.

For usual physical situations (assumed here), we have $c_{0,1} \propto \int d\theta f(\theta) \equiv 0$ [24]. In this case, $\Psi_0(q \to 0) = 1 + \mathcal{O}(q^2)$. In the general case, we have $\Psi_j(q \to 0) = \mathcal{O}(q^2)$ for $j \geq 1$. The integral (8) can be solved by the method of residues truncating the series at $j = N$ and after taking the limit $N \to \infty$. The trivial resonance $z = 1$ is related to the equilibrium state found for $m = m' = q = 0$. The nontrivial leading resonance can be evaluated by the well-known Newton-Raphson iterative method beginning with $z_0 = 1$ and converging to $z_\infty = \sum_{j=0}^{\infty}\Psi_j(q) + \mathcal{O}(q^4)$. In the limit $q \to 0$, this resonance will dominate the integral in the asymptotic limit $n \to \infty$. Thus, the evolution of the relevant Fourier coefficients can be written as $a_n(q) = \exp[n\gamma(q)]a_0(q)$, where the leading PR resonance is given by

$$\gamma(q) = \ln\sum_{j=0}^{\infty}\Psi_j(q) + \mathcal{O}(q^4). \quad (10)$$

From (10) the diffusion coefficient can be calculated as $D = -(1/2)[\partial_q^2\sum_{j=0}^{\infty}\Psi_j(q)]_{q=0}$. Applying this expression to the memory functions (9a)–(9c), the general exact diffusion coefficient formula will be given by

$$\frac{D}{D_{ql}} = 1 + 2\sum_{m=1}^{\infty}\sigma_{m,m}\mathrm{Re}[\mathcal{G}_0(r, mc)] + \sum_{j=2}^{\infty}\sum_{\{m\}}\sum_{\{\lambda\}^\dagger}\sigma_{m_1,m_j}\mathcal{G}_{\lambda_1}(r, m_1c) \times \prod_{i=2}^{j}\mathcal{G}_{\lambda_i}(r, m_ic)\mathcal{J}_{m_{i-1}-m_i}\left(-\frac{K}{r}\sum_{k=1}^{i-1}\lambda_k\right), \quad (11)$$

where $D_{ql} = -c_{0,2}K^2$ is the quasilinear diffusion coefficient and $\sigma_{m,m'} = (c_{-m,1}c_{m',1})/c_{0,2}$. The diffusion formula (11) assumes a more simple form for the LRN case (where I can be replaced by I mod 2π), yielding $\mathcal{G}_\lambda(1,x) = \delta_{\lambda,x}$ and

$$\frac{D_{\mathrm{LRN}}}{D_{ql}} = 1 + \sum_{j=2}^{\infty}\sum_{\{m\}^\dagger}\sigma_{m_1,m_j}\prod_{i=2}^{j}\mathcal{J}_{m_{i-1}-m_i}\left(-K\sum_{k=1}^{i-1}m_k\right). \quad (12)$$

As a check of this theory, we can first calculate D_{LRN} explicitly for two cases: (i) the well-known standard map (*sm*) as an example of a mixed system and (ii) a sawtooth map (*sw*) as an example of a hyperbolic system in a certain parameter regime. In case (i), we have $f(\theta) = \sin(\theta)$, and, hence, $\mathcal{J}_m(x)$ is the Bessel function of the first kind $J_m(x)$, $D_{ql} = K^2/4$, and $\sigma_{m,m'} = (\pm\delta_{m,\pm 1})(\pm\delta_{m',\pm 1})$. The resultant expression for D_{sm} is very similar to (12). The first terms of the expansion coincide with the Rechester, Rosenbluth, and White results [2]: $D_{sm}/D_{ql} = 1 - 2J_2(K) + 2J_2^2(K) + \cdots$. In case (ii), we have $f(\theta) = \theta$; hence, $\mathcal{J}_m(x) = \sin[\pi(m-x)]/\pi(m-x)$, $D_{ql} = K^2\pi^2/6$, and $\sigma_{m,m'} = (6/\pi^2)[(-1)^{m-m'}/mm']$. The sawtooth map is hyperbolic when $|K+2| > 2$ [25]. Finally, we also consider a third case where $f(x) = \alpha(x) = \sin(x)$, known as the *Harper map* (Hm), as an example of a map with a nonlinear rotation number. The resultant expression for D_{Hm} is very similar to (11), where $\mathcal{G}_\lambda(1, x) = J_\lambda(x)$ and other terms follow case (i). The analytical results of the three cases are compared with numerical calculations of D/D_{ql} in Figs. 1(a)–1(c). Despite the accelerator modes, whose kinetic properties are anomalous [26], all theoretical results are in excellent agreement with the numerical simulations.

A question of interest that arises here is the oscillatory character of the diffusion coefficient for maps with a periodic rotation number (including the LRN case), in contrast to the fast asymptotic behavior exhibited by maps with a nonperiodic rotation number (see, for example, [7]). The nonperiodic case can be considered by applying the limit $r \to \infty$. For such a case, we have $\lambda r^{-1} \to s$, $r^{-1}\sum_\lambda \to \int ds$, and $r\mathcal{G}_\lambda(r, x) \to \mathcal{G}(s, x)$ is the s-Fourier transform of $e^{-ix\alpha(I)}$. For cases where $\mathcal{G}(s, x)$ is well defined, the limit $r \to \infty$ produces high oscillatory integrals resulting in $D \to D_{ql}$ without any oscillation. In the case of a standard map, Chirikov [1] conjectured that the oscillatory aspect of the diffusion curve was an effect of the "islands of stability," but a satisfactory explanation of the oscillations has not been given yet [27].

Returning to Eq. (11), we can note that, in the limit of high stochasticity parameter K, the diffusion coefficient does not necessarily converge to the quasilinear value in the nonlinear rotation number cases. The standard argument in this respect is the so-called *random phase approximation* [1,19]. The intuitive idea is that, for large values of K, the phases $\theta_n(I, \theta)$ oscillate so fast that they become uncorrelated from θ. In order to verify this effect, we can take the limit $K \to \infty$ of (11) by setting $\lambda_i = 0$ for all i to avoid terms of order $\mathcal{O}(K^{-1/2})$. Once $|\mathcal{G}_0(r, mc)| < 1$ for $m \neq 0$, the asymptotic diffusion becomes a geometric sum whose result is

$$\lim_{K\to\infty} \frac{D}{D_{ql}} = 1 + \sum_{m\neq 0} \sigma_{m,m} \frac{\mathcal{G}_0(r, mc)}{1 - \mathcal{G}_0(r, mc)}. \qquad (13)$$

The rate (13) diverges at $c = 0$, creating a kind of accelerator mode. Indeed, a direct calculation through Eq. (1) shows that D/D_{ql} diverges as n in this case for all $K \neq 0$.

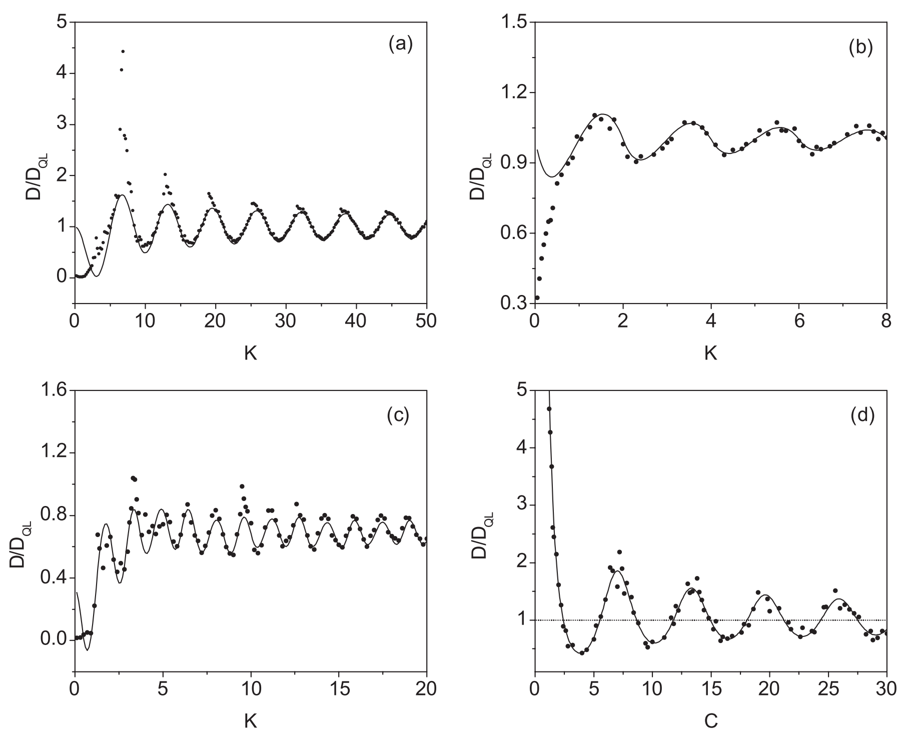

FIG. 1. Theoretical diffusion coefficient rate D/D_{ql} (solid lines) compared with numerical simulations calculated for $n = 100$. In (a), (b), and (c) we truncate the diffusion formulas (11) and (12) at $j = 2$. A better agreement for small values of K requires the calculation of further memory functions. (a) Standard map. The accelerator modes give rise to spikes in the figure. (b) Sawtooth map and (c) Harper map for $c = 5.5$ (with the presence of accelerator modes). (d) Harper map as a function of c for $K = 10^5$. The angular evolution induces fast and slow modes of diffusion even in the high stochasticity regime. This strong angular memory effect decays as $2J_0(c)/[1 - J_0(c)]$.

In Fig. 1(d), we consider the double sine map for $K = 10^5$. As one can see, even in the high stochasticity regime, where the random phase approximation is expected to hold, the rate D/D_{ql} oscillates between the zeros of $J_0(c)$. Its maximum and minimum values are ruled by zeros of $J_1(c)$. This strong angular memory effect is a remarkable result.

Another important question concerns the higher-order transport coefficients that play a central role in the large deviations theory. These coefficients can be obtained through the following dispersion relation:

$$\mathcal{D}_{2l} \equiv \lim_{n\to\infty} \frac{\langle (I_n - I_0)^{2l} \rangle_c}{(2l!)n} = \frac{(-1)^l}{(2l)!} \partial_q^{2l} \gamma(q)|_{q=0}, \quad (14)$$

where $l \geq 1$ and $\langle \rangle_c$ denotes cumulant moments [28]. The diffusion coefficient is defined by $D = \mathcal{D}_2$. The higher-order coefficients $\mathcal{D}_{2l}$ can be calculated by introducing successive corrections $\mathcal{O}(q^{2l})$ in (10). If the evolution process were asymptotically truly diffusive, then the angle-averaged density would have a Gaussian contour after a sufficiently long time. A first indication of the deviation of a density function from a Gaussian packet is given by the fourth-order Burnett coefficient $B \equiv \mathcal{D}_4$: If $B = 0$, then the kurtosis $\kappa(x) = \langle x^4 \rangle / \langle x^2 \rangle^2$ for $x = I_n - I_0$ is equal to 3 in the limit $n \to \infty$, a result valid for a Gaussian density for all times. These aspects will be treated elsewhere [29].

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