Critical Dynamics of the Dirty Boson Problem: Revisiting the Equality $z = d$

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It is shown that previous arguments, leading to the equality $z = d$ for the dynamical exponent describing the Bose glass to superfluid transition in d dimensions, may break down, as apparently seen in recent simulations. The key observation is that the major contribution to the compressibility, which remains finite through the transition and was predicted to scale as $\kappa \sim |\delta|^{(d-z)\nu}$ (where δ is the deviation from criticality and ν is the correlation length exponent) comes from the analytic, not the singular part of the free energy, and is not restricted by any conventional scaling hypothesis.

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Beginning with the realization that the $T = 0$ onset of superfluidity in a random medium should be treated as a fluctuation driven quantum phase transition [1–4], several scaling arguments were put forward [2,3] to restrict the critical exponents describing it. Most significant was an argument equating the dynamical exponent z, describing the relative divergence of the temporal and spatial correlation lengths via $\xi_\tau \sim \xi^z$, to the dimension of space d. The argument was supported by exact calculations in $d = 1$, where indeed $z = 1$ [3], by $1 + \epsilon$ renormalization group calculations [5] which, however, lack rigor due to the absence of a form for the Hamiltonian for noninteger $d > 1$, and by a series of quantum Monte Carlo (QMC) studies in $d \le 2$ [6,7]. Results in $d > 2$ are restricted to a badly controlled double ϵ-expansion which is inappropriate for testing an exact scaling argument of this type [4,8], but nevertheless points to values of z significantly larger than unity [8].

With one controversial exception [7], the earlier QMC results [6] were fit to $z = d$, but the numerical error bars were not very tight. The more recent QMC study in $d = 2$ [9] provided a much more stringent test: using larger systems and a joint, optimal critical functional form fit to three separate thermodynamic quantities, the value $z = 1.40 \pm 0.02$ was found, violating $z = d$. The purpose of this Letter is to revisit the scaling arguments [2,3] and show that they can in fact *break down:* z might be independent of all the other exponents. Absent further theoretical justification, we propose that z may be unconstrained by any simple scaling argument.

Before going into detail, we briefly review the essence of the scaling arguments and indicate where they could go wrong. The helicity modulus, Υ [or superfluid density, $\rho_s = (m^2/\hbar^2)\Upsilon$], quantifies the response of the superfluid to gradients in the phase, ϕ, of the order parameter: the free energy density contains a correction $\Delta f_x = (\Upsilon/2\beta V) \times \int_0^\beta d\tau \int d^d x |\nabla\phi(\mathbf{x},\tau)|^2$, where V is the volume, $\beta = 1/k_B T$, and τ is the usual imaginary time variable. The essence of the Josephson scaling argument is that since $\nabla\phi$ has dimensions of inverse length, it should scale as $\xi^{-1} \sim |\delta|^\nu$ where δ is the deviation from criticality and ν is the correlation length exponent. Since the singular part of the free energy, f_s, is defined to scale as $|\delta|^{2-\alpha}$, this implies that $\Upsilon \sim |\delta|^\upsilon$ with $\upsilon = 2 - \alpha - 2\nu = (d + z - 2)\nu$ (the last following from the quantum hyperscaling relation $2 - \alpha = (d + z)\nu$ [3]). Now, the Josephson relation, connecting changes in the chemical potential to the time derivative of ϕ, also allows one to interpret the *compressibility*, κ, as a helicity modulus in the imaginary time direction. Thus, κ enters a free energy correction $\Delta f_\tau = (\kappa/2\beta V) \times \int_0^\beta d\tau \int d^d x (\partial_\tau\phi)^2$. This suggests that $\partial_\tau\phi$ should scale as $\xi_\tau^{-1} \sim |\delta|^{z\nu}$, leading to $\kappa \sim |\delta|^{\upsilon_\tau}$ with $\upsilon_\tau = 2 - \alpha - 2z\nu = (d - z)\nu$. Since both the superfluid and the Bose glass phases have finite compressibility, one expects on physical grounds that κ should be finite and nonzero at $\delta = 0$ as well. This immediately requires $z = d$ [2,3]. However, a hidden assumption here is that Δf_x and Δf_τ are included in the singular part of the free energy, f_s. We shall show that this is correct for Δf_x but not for Δf_τ. In fact, the main contribution to Δf_τ comes from the analytic part of the free energy, f_a, so that κ is dominated by its analytic part which is trivially finite at the transition, and f_s yields only corrections that vanish at $\delta = 0$.

Rigorous definitions of helicity modulii compare free energies with twisted and untwisted boundary conditions. The boundary condition dependence, Δf, is normally included in f_s: the strong dependence required for a finite helicity modulus requires long or quasilong range order, present only in the superfluid. The fact that $\Upsilon \equiv 0$ in the disordered phase then guarantees that it can arise only from singular terms in the free energy. In the present problem, however, κ is nonzero in both phases: gapless excitations in the Bose glass phase lead to power law order in imaginary time (though not in space) [3]. The free energies of both, therefore, have strong temporal boundary condition dependence, and an analytic contribution is very natural.

From our analysis, there emerges the following general criterion: Δf should be included in f_s only if the twisted boundary condition breaks a fundamental symmetry of the model, since it can then be expected to generate relevant (in the renormalization group sense) terms in the Lagrangian, leading to a new diverging scaling variable in f_s dominating all other contributions to Δf. In this case, the usual Josephson scaling relation will hold. On the other hand, if no additional symmetry is broken, no new relevant scaling variable results, and the twist will lead only to small shifts in the parameters already present in the untwisted Lagrangian. The helicity modulus, which involves derivatives with respect to these shifts, will then be dominated by f_a. In the present problem, the twist couples to particle-hole symmetry [8], which is always broken at the Bose glass to superfluid transition.

For convenience, we consider a continuum $|\psi|^4$-model functional integral representation of the partition function, $Z = \int D\psi \exp(-\mathcal{L}_B)$, with Lagrangian [3,8]

$$\begin{aligned}\mathcal{L}_B = \int_0^\beta d\tau \int d^dx\{&-J\psi^*(\mathbf{x},\tau)\nabla^2\psi(\mathbf{x},\tau) \\ &- K\psi^*(\mathbf{x},\tau)[\partial_\tau - \mu(\mathbf{x})]^2\psi(\mathbf{x},\tau) \\ &+ r(\mathbf{x})|\psi(\mathbf{x},\tau)|^2 + u|\psi(\mathbf{x},\tau)|^4\},\end{aligned} \tag{1}$$

where $J \approx \hbar^2/2m > 0$ is the boson hopping amplitude favoring spatial ferromagnetic order in the phase of ψ, $\mu(\mathbf{x})$ is the (static) random external potential, and $K \approx 1/2u_0 > 0$, where u_0 is the soft core repulsion between nearby bosons, favors temporal ferromagnetic order as well. This model is a continuum approximation to the Josephson junction array Lagrangian in which $\psi(\mathbf{x},\tau) \rightarrow \exp[i\phi_i(\tau)]$ where $\phi_i(\tau)$ is the Josephson phase at site i, and $\psi^*\nabla^2\psi$ is the continuum limit of the Josephson coupling, $J_{ij}\cos[\phi_i(\tau) - \phi_j(\tau)]$, between two nearby sites i, j. The $r|\psi|^2 + u|\psi|^4$ terms represent the usual soft spin approximation to the constraint $|\psi| = 1$. We allow $r(\mathbf{x})$ to be random as well, representing disorder in the hopping amplitudes, J_{ij}. Disorder in J and u is also allowed, but is less convenient and produces nothing new. We write $\mu(\mathbf{x}) = \mu - \epsilon(\mathbf{x})$ and $r(\mathbf{x}) = r_0 + w(\mathbf{r})$ with disorder average $[\epsilon(\mathbf{x})]_{\rm av} = [w(\mathbf{x})]_{\rm av} = 0$. For given μ, the transition to superfluidity occurs with increasing J at a point $J_c(\mu)$, and we take $\delta = J - J_c(\mu)$.

Other representations serve equally well. The arguments we present are general and do not depend on the precise form of $\mathcal{L}_B$. The only common feature required is that time derivatives and chemical potential always appear in the combination $(\partial_\tau - \mu)$. In the coherent state functional integral formulation, for example, the Lagrangian has only a linear term, $\psi^*(\partial_\tau - \mu)\psi$. The two formulations match up to irrelevant terms (in the renormalization group sense) so long as $\mu(\mathbf{x})$ does not vanish identically [3,8]. When $\mu(\mathbf{x}) \equiv 0$ the model (1) reduces to the well known random rod problem, corresponding to a classical $(d+1)$-dimensional XY-model with columnar disorder [8,10,11]. This model retains an exact particle-hole symmetry, and the Josephson scaling relation for κ is, therefore, valid, but with $z < d$: see below.

Twisted θ-boundary conditions [12] are defined by the condition

$$\psi(x + L_\alpha \hat{\mathbf{e}}_\alpha) = e^{i\theta_\alpha}\psi(x), \qquad |\theta_\alpha| \le \pi, \tag{2}$$

where $x \equiv (\tau, \mathbf{x})$, $\hat{\mathbf{e}}_\alpha$, $\alpha = 0, 1, \ldots, d$, are space-time unit vectors, and L_α is the dimension of the system along $\hat{\mathbf{e}}_\alpha$ (with $L_0 = \beta$). Thus, $\bar{\psi}(\mathbf{x}) = e^{-i\mathbf{k}_0\cdot\mathbf{x} - i\omega_0\tau}\psi(\mathbf{x})$, where $\omega_0 = \theta_0/\beta$ and $\mathbf{k}_0 = (\theta_1/L_1, \ldots, \theta_d/L_d)$ obey periodic boundary conditions. A uniform $\bar{\psi}_0 = \langle\bar{\psi}(\mathbf{x})\rangle$ leads to an order parameter $\langle\psi(\mathbf{x})\rangle = e^{i\mathbf{k}_0\cdot\mathbf{x} + i\omega_0\tau}\bar{\psi}_0$ with uniform phase twist. The Lagrangian in terms of $\bar{\psi}$ is

$$\mathcal{L}_B^{\omega_0\mathbf{k}_0}[\psi;\mu,r_0] = \mathcal{L}_B[\bar{\psi};\mu - i\omega_0; r_0 + J\mathbf{k}_0^2] + \mathbf{k}_0\cdot\mathbf{P}[\bar{\psi}], \tag{3}$$

where $\mathbf{P}[\bar{\psi}] = -iJ\int d^dx \int_0^\beta d\tau[\bar{\psi}^*\nabla\bar{\psi} - \bar{\psi}\nabla\bar{\psi}^*]$ is the momentum. The shift, $\mu \rightarrow \mu - i\omega_0$, is guaranteed by the combination $(\partial_\tau - \mu)$ for any dirty boson model.

The free energy is $f^{\mathbf{k}_0\omega_0} = (\beta V)^{-1}[\ln(Z)]_{\rm av}$, labeled by the boundary condition. Using (3), Taylor expansion of the free energy in powers of $\mathbf{k}_0$ at $\omega_0 = 0$,

$$\Delta f_x \equiv f^{\mathbf{k}_0} - f = \frac{1}{2}\Upsilon\mathbf{k}_0^2 + O(\mathbf{k}_0^4), \tag{4}$$

produces the usual momentum-momentum type correlation function expression for Υ. On the other hand, if $\mathbf{k}_0 = 0$, one has $f^{\omega_0}(\mu) = f(\mu - i\omega_0)$. Analyticity within a given thermodynamic phase immediately implies

$$\Delta f_\tau \equiv f^{\omega_0} - f = i\omega_0\rho + \frac{1}{2}\kappa\omega_0^2 + O(\omega_0^3), \tag{5}$$

where $\rho = -\partial f/\partial\mu$ and $\kappa = \partial\rho/\partial\mu$. Comparison of (4) and (5) motivates identification of κ with the temporal helicity modulus. However, the density ρ actually yields the leading term, linear in ω_0. This term appears in (5) but not in (4) because $\mathcal{L}_B$ contains only quadratic spatial derivatives and is therefore even under space inversion, $\mathbf{x} \rightarrow -\mathbf{x}$. Since $\mathbf{P}[\psi]$ is odd under space inversion, $f^{\mathbf{k}_0\omega_0}$ must be an even function of $\mathbf{k}_0$. Moreover, $\mathbf{k}_0$ *breaks* the inversion symmetry of $\mathcal{L}_B[\psi]$. In contrast, only if $\mu(\mathbf{x}) \equiv 0$ does $\mathcal{L}_B$ possess inversion symmetry in time, $\tau \rightarrow -\tau$, which we call particle-hole symmetry [13]. This symmetry is already broken by μ, and the additional twist, ω_0, produces nothing new. These observations will be crucial to understanding the scaling of Υ and κ.

Let us consider first the Josephson scaling argument for the random rod problem, which possesses both inversion and particle-hole symmetry. Classical intuition suggests that $\kappa \equiv 0$ in the disordered phase [14], and the leading boundary condition dependence must enter via singular scaling combinations $k_0\xi$ and $\omega_0\xi_\tau$. This is formally implemented via a finite-size scaling ansatz [3,12],

$$\Delta f^{\mathbf{k}_0\omega_0} \approx \beta^{-1}L^{-d}\Phi_0^{\mathbf{k}_0\omega_0}(A_0\delta L^{1/\nu_0}, B_0\delta\beta^{1/z_0\nu_0}), \tag{6}$$

where A_0 and B_0 are nonuniversal scale factors [15]. For convenience, we take a hypercubical volume $V = L^d$, and the exponents are those of the classical random rod problem. The existence of a nonzero stiffness, i.e., a leading finite-size correction of order L^{-2} or β^{-2}, requires that the scaling function $\Phi_0^{\mathbf{k}_0\omega_0}(x, y) \approx x^{d\nu_0} y^{z_0\nu_0}(Q_x x^{-2\nu_0} + Q_\tau y^{-2z_0\nu_0})$ for large x, $y > 0$, yielding

$$\begin{aligned} \Upsilon &\approx A_0^{(d-2)\nu_0} B_0^{z_0\nu_0}(Q_x/\theta_x^2)\delta^{\upsilon_0} \\ \kappa &\approx A_0^{d\nu_0} B_0^{-z_0\nu_0}(Q_0/\theta_0^2)\delta^{\upsilon_{\tau 0}}, \end{aligned} \tag{7}$$

where $\theta_x^2 = \theta_1^2 + \ldots + \theta_d^2$, implying Josephson scaling $\upsilon_0 = (d + z_0 - 2)\nu_0 = 2 - \alpha_0 - 2\nu_0$, $\upsilon_{\tau 0} = (d - z_0)\nu_0 = 2 - \alpha_0 - 2z_0\nu_0$, and in addition $Q_{x,0} \propto \theta_{x,0}^2$.

Consider now the Bose glass to superfluid transition. For both phases, ρ and κ are smooth nontrivial functions of μ, and a temporal twist now perturbs only slightly the particle-hole symmetry breaking term already in $\mathcal{L}_B$. On the other hand, spatial twists still produce singular corrections, and *at* $\omega_0 = 0$ the scaling form [16]

$$\Delta f^{\mathbf{k}_0} = \beta^{-1} L^{-d} \Phi^{\mathbf{k}_0}(A\delta L^{1/\nu}, B\delta\beta^{1/z\nu}), \tag{8}$$

is predicted, with $\Phi^{\mathbf{k}_0}(x, y) \approx R_x x^{(d-2)\nu} y^{z\nu}$ for large x, $y > 0$, yielding $\Upsilon \approx A^{(d-2)\nu} B^{z\nu}(R_x/\theta_x^2)\delta^{\upsilon}$, $\upsilon = (d + z - 2)\nu = 2 - \alpha - 2\nu$, and $R_x \propto \theta_x^2$ as before. All exponents now refer to the dirty boson critical point. Because we have, as yet, imposed no temporal twist, there can be no $O(\beta^{-1}, \beta^{-2})$ terms.

Now, if a finite ω_0 is included, the only changes in (8) are that $\mu \to \mu - i\omega_0$ everywhere, and boundary condition dependence of f_a must be included. Thus

$$\begin{aligned} \Delta f^{\mathbf{k}_0\omega_0} = &\beta^{-1} L^{-d} \Phi^{\mathbf{k}_0}(A\delta_\theta L^{1/\nu}, B\delta_\theta \beta^{1/z\nu}) \\ &+ f_a(J, r_0 + J\mathbf{k}_0^2, \mu - i\omega_0) - f_a(J, r_0, \mu), \end{aligned} \tag{9}$$

where $\delta_\theta = J - J_c(\mu - i\omega_0) \approx \delta + i\omega_0 J_c'(\mu)$. Most importantly, $\Phi^{\mathbf{k}_0}$ *is the same function* as that in (8), and can still yield no $O(\beta^{-1}, \beta^{-2})$ terms. This implies that $\Phi^{\mathbf{k}_0\omega_0}$ produces no *direct* contributions to ρ and κ, which must therefore arise (a) *indirectly* from the ω_0 dependence of δ_θ and (b) directly from the analytic part of the free energy. The former couples derivatives with respect to μ (equivalently, ω_0) to those with respect to δ, producing the leading singular terms, $\kappa_s \sim |\delta|^{-\alpha}$ [17]. However, $\alpha = 2 - (d + z)\nu$ is very likely negative [3], so this gives a vanishing contribution at $\delta = 0$, and the main contribution is analytic in origin. Taking $\mathbf{k}_0 = 0$, we may write $f_a(J, \mu) = -\rho_c(J)[\mu - \mu_c(J)] - \frac{1}{2}\kappa_c(J)[\mu - \mu_c(J)]^2 + \ldots,$ expanded for convenience about the transition line $\mu_c(J)$, and we obtain from (9) a finite κ through the transition, with exponent z nevertheless *undetermined* [18,19].

A second approach to the derivation of the Josephson scaling relation, via scaling of the two-point correlation function in the superfluid hydrodynamic regime, is now discussed. Long wavelength, low frequency fluctuations are governed by the effective Gaussian Lagrangian [2,3]

$$\mathcal{L}_{\rm HD} = \frac{1}{2}\int d^d x \int_0^\beta d\tau[\Upsilon|\nabla\tilde{\phi}|^2 + \kappa(\partial_\tau\tilde{\phi})^2], \tag{10}$$

where $\tilde{\phi}(\mathbf{x}, \tau)$ is the coarse-grained phase, related to the coarse-grained order parameter field via $\tilde{\psi}(\mathbf{x}, \tau) = \psi_0 \exp[i\tilde{\phi}(\mathbf{x}, \tau)]$, with bulk order parameter, $\psi_0 \sim |\delta|^\beta$, near criticality. The small $\mathbf{k}$, ω form of the Fourier transform of the two-point correlator, $G(\mathbf{x} - \mathbf{x}', \tau - \tau') = [\langle\psi^*(\mathbf{x}, \tau)\psi(\mathbf{x}', \tau')\rangle]_{\rm av}$, is then

$$G(\mathbf{k}, \omega) \approx |\psi_0|^2/[\Upsilon\mathbf{k}^2 + \kappa\omega^2]. \tag{11}$$

Normally, it is assumed that G obeys the scaling form

$$G(\mathbf{k}, \omega) \approx C|\delta|^{-\gamma} g(Dk|\delta|^{-\nu}, E\omega|\delta|^{-z\nu}), \tag{12}$$

where γ is the susceptibility exponent and C, D, E are nonuniversal scale factors. If one naively matches (11) and (12), one concludes that $g(x, y) \approx (g_1 x^2 + g_2 y^2)^{-1}$ for small x, y, and hence that

$$\begin{aligned} \Upsilon/|\psi_0|^2 &\approx g_1 C^{-1} D^2 |\delta|^{\gamma - 2\nu} \\ \kappa/|\psi_0|^2 &\approx g_2 C^{-1} E^2 |\delta|^{\gamma - 2z\nu}. \end{aligned} \tag{13}$$

Using the scaling relation $\alpha + 2\beta + \gamma = 2$, we obtain again the relations $\upsilon = 2 - \alpha - 2\nu$, $\upsilon_\tau = 2 - \alpha - 2z\nu$.

However, this is just a disguised version of the free energy argument: $\mathcal{L}_{\rm HD}$ assumes that the energetics of global phase twists also describes slowly varying local phase twists. Thus, locally we replace $\mathbf{k}_0$ by $\nabla\tilde{\phi}$ and ω_0 by $\partial_\tau\tilde{\phi}$, then integrate over space-time [20]. Since κ arises from the nonscaling part of the free energy, it is unlikely that it can now arise from the scaling part of the two-point function. Rather, we must carefully reconsider the scaling ansatz for G. Our proposal (for which we have no detailed theoretical support at this stage), is that

$$\begin{aligned} |\psi_0|^2/G(\mathbf{k}, \omega) \approx &H|\delta|^{2-\alpha}\Gamma(Dk|\delta|^{-\nu}, E\omega|\delta|^{-z\nu}) \\ &+ \Gamma_a(\mathbf{k}, \omega), \end{aligned} \tag{14}$$

where Γ_a is analytic. This self-energy scaling is very similar to (9). Matching with (11), we again assume that Υ arises from the scaling part, κ from the analytic part. Thus $\Gamma(x, y) \approx \gamma_1 x^2 + \gamma_2 y^2$, for small x, y while $\Gamma_a(k, \omega) \approx \kappa\omega^2$ for small k, ω. The x^2 term yields $\Upsilon \sim |\delta|^\upsilon$. If $z < d$, the y^2 term is subdominant to the analytic term, κ is finite, as required, and z is undetermined [21,22]. Standard static scaling is recovered, without any unusual analytic corrections, for $\omega = 0$.

To summarize, the original scaling ansatz for Bose glass-superfluid criticality, with temporal twists producing a relevant symmetry breaking perturbation to the Lagrangian, scaling in the combination $\omega_0\xi_\tau$, is not supported theoretically, and may explain violations of $z = d$ in $d = 2$ [9].

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[12] M. E. Fisher, M. N. Barber, and D. Jasnow, Phys. Rev. A **8**, 1111 (1973). Amazingly, the last section of this paper, containing a discussion of how finite-size scaling might lead to violations of the Josephson relation, proposes a mechanism very similar to ours: anomalously strong boundary condition dependence of t_θ, the distance from the critical point, on the twist wave vector, k_0. If $t_\theta - t \sim k_0^2$, they find $\upsilon = 1 - \alpha$. Here we find an even more anomalous linear dependence of $\delta_\theta - \delta$ on ω_0, leading to $\upsilon_\tau = -\alpha$. We are unaware of any classical model where the original proposed violation occurs.

[13] This must be contrasted with time reversal symmetry, which involves a simultaneous replacement $\psi \leftrightarrow \psi^*$, which $\mathcal{L}_B$ always satisfies.

[14] This can be demonstrated more explicitly: P. B. Weichman and R. Mukhopadhyay (unpublished).

[15] No overall multiplicative scale factor is required due to quantum hyperuniversality: see K. Kim and P. B. Weichman, Phys. Rev. B **43**, 13583 (1991).

[16] In Ref. [8], it was argued that so long as $\epsilon(\mathbf{x}) \neq 0$, μ itself was irrelevant; i.e., a statistical particle-hole symmetry is restored at the critical point. This would imply a correction to scaling combination $\mu|\delta|^{\lambda_\mu}$, $\lambda_\mu > 0$ which we have ignored in (8). This means that μ drops out completely sufficiently close to the critical point, ruling out again any scaling of chemical potential changes with ξ_τ.

[17] Within the ϵ, ϵ_τ-expansion [10], where ϵ_τ is the dimension of time and $\epsilon = 4 - d - \epsilon_\tau$, it was found [8] that the Bose glass phase is compressible *only* in the physical dimension, $\epsilon_\tau = 1$, with $\kappa \equiv 0$ for any $\epsilon_\tau < 1$. This precludes any analytic contribution to κ. We expect then that $\kappa \sim |\delta|^{-\alpha(\epsilon,\epsilon_\tau)} \rightarrow 0$ from the superfluid side.

[18] The fact that the leading correction is *linear* in ω_0 leads even more directly to problems with the previous scaling ansatz [2,3]: if a finite $\kappa = \partial\rho/\partial\mu$ is to be associated with the singular part of the free energy, it must come from the singular part of the density which, for consistency, must scale as $\xi^{-d} \sim |\delta|^{d\nu}$. This vanishes rapidly at the critical point, contradicting the original assumption that the transition takes place at finite boson density.

[19] This can be confirmed explicitly in $d = 1$ [14]. Except for rapidly decaying irrelevant terms, μ can be removed entirely from the dual surface roughening model Lagrangian via a shift, appearing, consistent with our general arguments, only as an additive analytic term in the free energy. The result $z = 1$ for this model is therefore unconnected to the appearance of a finite κ.

[20] The linear term, $i\rho(\partial_\tau\tilde{\phi})$, integrates to an integer multiple of $2\pi i\rho$ and disappears from $\exp(-\mathcal{L}_{\mathrm{HD}})$ since the total boson number ρV is an integer.

[21] There are dynamical scaling breakdown issues at the finite T lambda transition (described by the classical Model F equations) as well, where the finite κ argument now yields $z = d/2$. However, violations are possible where there exist two different dynamical exponents z_1, z_2 satisfying $z_1 + z_2 = d$. Only the mean $(z_1 + z_2)/2 = d/2$ enters the corresponding hydrodynamic correlation function [V. Dohm, Phys. Rev. B **44**, 2697 (1991)].

[22] In a perhaps related effect, the rigorous finite-size correlation length exponent inequality $\nu_{\mathrm{FS}} \geq 2/d$ for classical random systems [J. T. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986)]does not preclude a second ''intrinsic'' exponent ν with $\nu < 2/d$ [F. Pázmándi, R. Scalettar, and G. T. Zimányi, Phys. Rev. Lett. **79**, 5130 (1997)]