Tensor Modes from a Primordial Hagedorn Phase of String Cosmology

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It has recently been shown that a Hagedorn phase of string gas cosmology can provide a causal mechanism for generating a nearly scale-invariant spectrum of scalar metric fluctuations, without the need for an intervening period of de Sitter expansion. In this Letter, we compute the spectrum of tensor metric fluctuations (gravitational waves) in this scenario and show that it is also nearly scale invariant. However, whereas the spectrum of scalar modes has a small red tilt, the spectrum of tensor modes has a small blue tilt, unlike what occurs in slow-roll inflation. This provides a possible observational way to distinguish between our cosmological scenario and conventional slow-roll inflation.

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Introduction.—String gas cosmology (SGC) [1,2] (see also [3] for early work and [4–6] for recent overviews) is an early approach to string cosmology, based on adding minimal but crucial inputs from string theory, namely, new degrees of freedom—string oscillatory and winding modes—and new symmetries—T duality—to the hypothesis of a hot and small early Universe. A key aspect of SGC is that the temperature cannot exceed a limiting temperature, the Hagedorn temperature T_H [7]. This provides a qualitative reason to expect that string theory can resolve cosmological singularities [1]. If we evolve the radiation-dominated Friedmann-Robertson-Walker (FRW) phase of standard cosmology into the past, a smooth transition to a quasistatic Hagedorn phase will occur. Reversing the time direction in this argument, the following new cosmological scenario [1] emerges: The Universe starts in a quasistatic Hagedorn phase, during which thermal equilibrium can be established over a large scale (large enough for our current Universe to grow out of it following the usual noninflationary cosmological dynamics). The quasistatic phase makes a smooth transition to a radiation-dominated FRW phase, after which point the Universe evolves as in standard cosmology.

Recently [8], it was discovered that string thermodynamic fluctuations during the Hagedorn phase lead to scalar metric fluctuations which are adiabatic and nearly scale invariant at late times, provided that the perturbations can be described by Einstein gravity, thus providing a simple alternative to slow-roll inflation for establishing such perturbations. Note that this mechanism for the generation of the primordial perturbations is intrinsically stringy—particle thermodynamic fluctuations would lead to a spectrum with a large and phenomenologically unacceptable blue tilt.

We briefly recall the key features of the new structure formation scenario. At early times t ($t < t_R$), the Universe is in the quasistatic Hagedorn phase. The physical wavelength of any perturbation mode (characterized by having constant momentum k in comoving coordinates) is approximately constant. The key point is that the Hubble radius (which sets the limit on which causal processes can locally set up fluctuations—see, e.g., [9,10] for overviews of the theory of cosmological perturbations) is essentially infinite, thus allowing a causal mechanism for the generation of primordial fluctuations. Near $t = t_R$, a smooth transition from the Hagedorn phase to the radiation-dominated phase of standard cosmology occurs. The Hubble radius decreases rapidly to take on a minimal value, which is microscopic (set by the temperature at $t = t_R$, which will be close to T_H). Thus, fluctuation modes of relevance to current cosmological observations exit the Hubble radius at times $t_i(k)$ close to t_R. During the radiation-dominated FRW phase, the Hubble radius increases linearly in t, while the physical wavelength of a perturbation mode grows only as $t^{1/2}$. Thus, at late times $t_f(k)$, the modes reenter the Hubble radius. Since the primordial perturbations in our scenario are of thermal origin (and there are no nonvanishing chemical potentials), they will be adiabatic, and, since they propagate on super-Hubble scales for a long time during the FRW phase, they will be squeezed and will lead to the same type of acoustic oscillations in the angular power spectrum of the cosmic microwave background anisotropies as what is produced in slow-roll inflation models.

In this Letter, we generalize our previous analysis [8] to allow us to compute, in addition to the scalar metric fluctuations (the metric perturbation modes which couple to matter), the tensor modes (gravitational waves). We find that the resulting spectrum of tensor modes is also nearly scale invariant but that it has a slightly blue tilt, unlike what happens in slow-roll inflation, where the tilt for the gravitational wave spectrum is also red. The scalar to tensor ratio is calculable from the detailed dynamics of the system (we postpone this calculation to a follow-up paper). It is set by how close the temperature is to the Hagedorn temperature when the scales which are measured today exit the Hubble radius at the end of the quasistatic Hagedorn phase. The predicted blue tilt of the spectrum of gravitational waves would allow us, in principle, to distinguish our scenario from the usual slow-roll inflationary models.

We begin this Letter with a recap of the method of Refs. [8,11] to compute mass fluctuations in the Hagedorn phase of SGC and generalize the method to yield fluctuations of arbitrary components of the energy-momentum tensor of the string gas. Then we relate the spectrum of gravitational waves in late time cosmology to the fluctuations of the energy-momentum tensor in the Hagedorn phase and show that the resulting power spectrum is nearly scale invariant.

In the following, we assume that our three spatial dimensions are already large during the Hagedorn phase (for a possible mechanism to achieve this, see [12]), while the extra spatial dimensions are confined to the string scale. For a mechanism to achieve this in the context of SGC, see [1] (see, however, [13,14] for caveats), and, for a natural dynamical mechanism arising from SGC to stabilize all of the moduli associated with the extra spatial dimensions, see [15–20]. To be specific, we take our three dimensions to be toroidal. The existence of one cycle results in the stability of string winding modes—and this is a key ingredient in our calculations.

Energy-momentum tensor correlation functions for closed strings.—The mean energy-momentum tensor $\langle T^\mu_\nu \rangle$ is given in terms of the thermal canonical partition function Z by

$$\langle T^\mu_\nu \rangle = 2\frac{G^{\mu\alpha}}{\sqrt{-G}}\frac{\partial \ln Z}{\partial G^{\alpha\nu}} \equiv \mathcal{D}^\mu_\nu \ln Z, \tag{1}$$

where $G_{\mu\nu}$ is the Euclidean metric of space-time [the time coordinate is compactified to a circle of radius β (the inverse temperature)].

Our aim is to calculate the fluctuations of $T_{\mu\nu}$ on various length scales R. For each R, we take our spatial coordinates to run over a fixed interval, e.g., $[0, 2\pi]$. Thus, the metric is given by

$$G_{\mu\nu} = \mathrm{diag}[\beta^2, R^2, R^2, R^2]. \tag{2}$$

Letting another derivative operator $\mathcal{D}$ act on $\ln Z$ gives two terms: one for which both derivative operators act on Z and a second which will contain the product of terms where one $\mathcal{D}$ acts on Z. Symmetrizing over the indices, we find the mean square fluctuations to be

$$C^{\mu\sigma}_{\nu\lambda} = \langle \delta T^\mu_\nu \delta T^\sigma_\lambda \rangle = \langle T^\mu_\nu T^\sigma_\lambda \rangle - \langle T^\mu_\nu \rangle \langle T^\sigma_\lambda \rangle = 2\frac{G^{\mu\alpha}}{\sqrt{-G}}\frac{\partial}{\partial G^{\alpha\nu}}\left(\frac{G^{\sigma\delta}}{\sqrt{-G}}\frac{\partial \ln Z}{\partial G^{\delta\lambda}}\right) + 2\frac{G^{\sigma\alpha}}{\sqrt{-G}}\frac{\partial}{\partial G^{\alpha\lambda}}\left(\frac{G^{\mu\delta}}{\sqrt{-G}}\frac{\partial \ln Z}{\partial G^{\delta\nu}}\right), \tag{3}$$

with $\delta T^\mu_\nu = T^\mu_\nu - \langle T^\mu_\nu \rangle$.

The partition function $Z = \exp(-\beta F)$ is given by the string free energy F. Thus, the string thermodynamical fluctuation in the energy density, denoted by the correlation function C^{00}_{00}, becomes

$$C^{00}_{00} = \langle \delta\rho^2 \rangle = \langle \rho^2 \rangle - \langle \rho \rangle^2 = -\frac{1}{R^6}\frac{\partial}{\partial\beta}\left(F + \beta\frac{\partial F}{\partial \beta}\right) = \frac{T^2}{R^6}C_V, \tag{4}$$

where $C_V = (\partial\langle E\rangle/\partial T)$, with $E \equiv F + \beta(\partial F/\partial\beta)$ and $V = R^3$ is the volume of three compact but large spatial dimensions. The fluctuation in spatial diagonal parts of the energy-momentum tensor is

$$C^{ii}_{ii} = \langle \delta T^{i2}_i \rangle = \langle T^{i2}_i \rangle - \langle T^i_i \rangle^2 = \frac{1}{\beta R^3}\frac{\partial}{\partial \ln R}\left(-\frac{1}{R^3}\frac{\partial F}{\partial \ln R}\right) = \frac{1}{\beta R^2}\frac{\partial p}{\partial R}, \tag{5}$$

with no summation on i. The pressure p is

$$p \equiv -V^{-1}(\partial F/\partial \ln R) = T(\partial S/\partial V)_E. \tag{6}$$

We now compute the correlation functions (4) and (5) using tools from string statistical mechanics. We follow the discussion in Ref. [21]. The starting point is the formula $S(E, R) = \ln\Omega(E, R)$ for the entropy in terms of $\Omega(E, R)$, the density of states. The density of states of a gas of closed strings on a large three-dimensional torus (with the radii of all internal dimensions at the string scale) is given by (see [21])

$$\Omega(E, R) \simeq \beta_H e^{\beta_H E + n_H V}[1 + \delta\Omega_{(1)}(E, R)], \tag{7}$$

where $\delta\Omega_{(1)}$ comes from the contribution to the density of states [when writing the density of states as a Laplace transform of $Z(\beta)$, which involves integration over β] from the closest singularity point β_1 to $\beta_H = (1/T_H)$ in the complex β plane. Note that $\beta_1 < \beta_H$, and β_1 is real. From Ref. [21], we have

$$\delta\Omega_{(1)}(E, R) = -\frac{(\beta_H E)^5}{5!}e^{-(\beta_H - \beta_1)(E - \rho_H V)}. \tag{8}$$

In the above, n_H is a (constant) number density of order ℓ_s^{-3} (ℓ_s is the string length), ρ_H is the "Hagedorn energy density" of order ℓ_s^{-4}, and

$$\beta_H - \beta_1 \sim \begin{cases} (\ell_s^3/R^2) & \text{for } R \gg \ell_s, \\ (R^2/\ell_s) & \text{for } R \ll \ell_s. \end{cases} \tag{9}$$

To ensure the validity of Eq. (7), we require $\delta\Omega_{(1)} \ll 1$ by assuming $\rho \equiv (E/V) \gg \rho_H$.

Combining the above, the entropy of the string gas in the Hagedorn phase becomes

$$S(E,R) \simeq \beta_H E + n_H V + \ln[1 + \delta\Omega_{(1)}], \tag{10}$$

and therefore the temperature $T(E,R) \equiv [(\partial S/\partial E)_V]^{-1}$ will be

$$T \simeq \left(\beta_H + \frac{\partial \delta\Omega_{(1)}/\partial E}{1 + \delta\Omega_{(1)}}\right)^{-1} \simeq T_H\left(1 + \frac{\beta_H - \beta_1}{\beta_H}\delta\Omega_{(1)}\right). \tag{11}$$

Using this relation, we can express $\delta\Omega_{(1)}$ in terms of T and R via

$$\ell_s^3 \delta\Omega_{(1)} \simeq -\frac{R^2}{T_H}\left(1 - \frac{T}{T_H}\right). \tag{12}$$

In addition, we find

$$\langle E \rangle \simeq \ell_s^{-3} R^2 \ln\left[\frac{\ell_s^3 T}{R^2(1 - T/T_H)}\right]. \tag{13}$$

Note that, to ensure $\delta\Omega_{(1)} \ll 1$ and $\langle E \rangle \gg \rho_H R^3$, we require $(1 - T/T_H) \ll (\ell_s^2/R^2)$.

The results (10) and (12) determine the correlation functions (4) and (5). To compute (4), note that

$$C_V \equiv -\left[T^2\left(\frac{\partial^2 S(E,R)}{\partial E^2}\right)_V\right]^{-1} \approx \frac{R^2/\ell^3}{T(1 - T/T_H)}, \tag{14}$$

which leads to

$$C_{00}^{00} = \langle \delta\rho^2 \rangle \simeq \frac{T}{\ell_s^3(1 - T/T_H)} \frac{1}{R^4}. \tag{15}$$

Note that the factor $(1 - T/T_H)$ in the denominator is responsible for giving the spectrum a slight red tilt. It comes from the differentiation with respect to T.

Evaluating (6),

$$p(E,R) \approx n_H T_H - \frac{2}{3}\frac{(1 - T/T_H)}{\ell_s^3 R} \ln\left[\frac{\ell_s^3 T}{R^2(1 - T/T_H)}\right] \tag{16}$$

immediately yields

$$C_{ii}^{ii} \simeq \frac{T(1 - T/T_H)}{\ell_s^3 R^4} \ln^2\left[\frac{R^2}{\ell_s^2}(1 - T/T_H)\right]. \tag{17}$$

Note that, since no temperature derivative is taken, the factor $(1 - T/T_H)$ remains in the numerator. This will lead to the slight blue tilt of the spectrum of gravitational waves.

Tensor modes from Hagedorn fluctuations.—We now estimate the dimensionless power spectrum of gravitational waves.

In slow-roll inflation, to leading order in perturbation theory, matter fluctuations do not couple to tensor modes. This is due to the fact that the matter background field is slowly evolving in time and the leading order gravitational fluctuations are linear in the matter fluctuations. In our case, the background is not evolving (at least at the level of our computations), and hence the dominant metric fluctuations are quadratic in the matter field fluctuations. At this level, matter fluctuations induce both scalar and tensor metric fluctuations. Thus, we expect that, in our string gas cosmology scenario, the ratio of tensor to scalar metric fluctuations will be larger than in simple slow-roll inflationary models.

We will extract the amplitude of the gravitational wave spectrum from the spatial fluctuations C_{jj}^{ii} of the energy-momentum tensor. Strictly speaking, it is the off-diagonal components which will couple uniquely to the tensor modes. We will estimate their order of magnitude by the order of magnitude of the diagonal terms computed in the previous section. This gives a good approximation, as can be checked by letting the metric in (2) depend on three separate scales R_i (where the index i runs from 1 to 3) and by extracting the off-diagonal correlation functions following the method of the previous section but taking mixed spatial derivatives.

Tensor perturbations in a spatially flat FRW universe take the form

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j. \tag{18}$$

Since to linear order in h_{ij} the Einstein tensor for fluctuations on a scale k is proportional to $k^2 h_{ij}(k)$, it follows from the space-space Einstein equations that

$$k^2 h_{ij}(k) \sim (8\pi G)\delta T_{ij}(k). \tag{19}$$

The power spectrum of the right-hand side of the above equation is given by the correlation function C_{jj}^{ii}. Thus, the dimensionless gravitational wave power spectrum is given by this correlation function. Therefore, from (19) one can calculate the dimensionless power spectrum for $h_k^{\pm}$, where $h_k^{\pm}$ is the amplitude of either of the two gravitational wave polarization modes. Dropping the superscript $\pm$ (due to symmetry, both polarization modes will be equally excited), we obtain

$$k^3|h(k)|^2 \sim k^{-4}(8\pi G)^2 C_{jj}^{ii}. \tag{20}$$

Inserting the result (5) for the correlation function yields

$$k^3|h(k)|^2 \sim \frac{64\pi^2 G^2 T}{\ell_s^3}(1 - T/T_H)\ln^2\left[\frac{1}{\ell_s^2 k^2}(1 - T/T_H)\right], \tag{21}$$

which, for temperatures close to the Hagedorn value, reduces to

$$k^3|h(k)|^2 \sim \left(\frac{\ell_{\rm Pl}}{\ell_s}\right)^4 (1 - T/T_H)\ln^2\left[\frac{1}{\ell_s^2 k^2}(1 - T/T_H)\right]. \tag{22}$$

This shows that the spectrum of tensor modes is—to a first approximation—scale invariant.

Discussion.—Our result (22) for the power spectrum of gravitational waves should be compared to the result for the power spectrum of scalar metric fluctuations computed in Ref. [8]:

$$\mathcal{P}_{\Phi}(k) \sim \left(\frac{\ell_{\rm pl}}{\ell_s}\right)^4 \frac{1}{1 - T/T_H}. \tag{23}$$

Note that, for a fixed scale k, both (22) and (23) must be evaluated at the time $t_i(k)$ when the mode k exits the Hubble radius at the end of the Hagedorn phase. Since $t_i(k)$ increases slightly as k increases, the temperature $T[t_i(k)]$ will be slowly decreasing. Hence, the expression in front of the logarithm in our final expression (22) for the power spectrum of tensor fluctuations yields a slight blue tilt. Values of k for which the perturbative analysis of string gas cosmology is consistent are on the high k side of the zero of the logarithm in (22)—this follows from the condition $|\delta\Omega_{(1)}| \ll 1$ and (12). Hence, the logarithmic factor in (22) adds to the blue tilt of the spectrum.

A heuristic way of understanding the origin of the slight blue tilt in the spectrum of tensor modes is as follows. The closer we get to the Hagedorn temperature, the more the thermal bath is dominated by long string states, and thus the smaller the pressure will be compared to the pressure of a pure radiation bath. Since the pressure terms (strictly speaking, the anisotropic pressure terms) in the energy-momentum tensor are responsible for the tensor modes, we conclude that the smaller the value of the wave number k [and thus the higher the temperature $T[t_i(k)]$] when the mode exits the Hubble radius, the lower the amplitude of the tensor modes. In contrast, the scalar modes are determined by the energy density, which increases at $T[t_i(k)]$ as k decreases, leading to a slight red tilt.

Comparing (22) and (23), we see that the tensor to scalar ratio r evaluated on a scale k is given by

$$r \sim \{1 - T[t_i(k)]/T_H\}^2 \ln^2\left[\frac{1}{\ell_s^2 k^2}\{1 - T[t_i(k)]/T_H\}\right]. \tag{24}$$

In principle (if the dynamical evolution from the Hagedorn phase to the radiation-dominated FRW phase were under complete analytical control), this quantity is calculable. If the string length were known, the factor $(1 - T/T_H)$ could be determined from the normalization of the power spectrum of scalar metric fluctuations. Since the string length is expected to be a couple of orders larger than the Planck length, the above factor does not need to be very small. Thus, a ratio r larger than in simple roll inflationary models may emerge.

Based on the results of this Letter, it thus appears promising that our scenario will give rise to testable predictions.

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