Detection Loophole in Asymmetric Bell Experiments

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The problem of closing the detection loophole with asymmetric systems, such as entangled atom-photon pairs, is addressed. We show that, for the Bell inequality I_{3322}, a minimal detection efficiency of 43% can be tolerated for one of the particles, if the other one is always detected. We also study the influence of noise and discuss the prospects of experimental implementation.

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Nonlocality is one of the most striking properties of quantum mechanics. Two distant observers, each holding half of an entangled quantum state and performing appropriate measurements, share correlations which are nonlocal, in the sense that they violate a Bell inequality [1]. In other words, those correlations cannot be reproduced by any local hidden variable (LHV) model. All laboratory experiments to date have confirmed quantum nonlocality [2–8]. There is thus strong evidence that nature is nonlocal. However, considering the importance of such a statement, it is crucial to perform an experiment free of any loopholes, which is still missing today. Another motivation comes from quantum information science, where the security of some quantum communication protocols is based on the loophole-free violation of Bell inequalities [9].

Performing a loophole-free Bell test is quite challenging. One first has to ensure that no signal can be transmitted from one particle to the other during the measurement process. Thus the measurement choice on one side and the measurement result on the other side should be spacelike separated. If this is not the case, one particle could send some information about the measurement setting it experiences to the other particle. This is the locality loophole [10]. Secondly the particles must be detected with a high enough probability. If the detection efficiency is too low, a LHV model can reproduce the quantum correlations. In this picture a hidden variable affects the probability that the particle is detected depending on the measurement setting chosen by the observer. This is the detection loophole [11,12].

In practice, photon experiments have been able to close the locality loophole [3–7]. However the optical detection efficiencies are still too low to close the detection loophole. For the Clauser-Horne-Shimony-Holt (CHSH) [13] inequality, an efficiency larger than 82.8% is required to close the detection loophole with maximally entangled states. Surprisingly, Eberhard [14] showed that this threshold efficiency can be lowered to 66.7% by using nonmaximally entangled states. Threshold efficiencies for other Bell inequalities have also been studied [15–17]. On the other hand, an experiment carried out on trapped ions [8] closed the detection loophole, but the ions were only a few micrometers apart. It would already be a significant step forward to close the detection loophole for well-separated systems. Recently, new proposals for closing both loopholes in a single experiment were reported [18,19].

In this Letter we focus on asymmetric setups, where the two particles are detected with different probabilities. This is the case, e.g., in an atom-photon system: the atom is measured with an efficiency close to 1 while the probability to detect the photon is smaller. Intuition suggests that if one party can do very efficient measurements, then the minimal detection efficiency on the other side should be considerably lowered compared to the case where both detectors have the same efficiency. Experimentally this approach might be quite promising, since recent experiments have demonstrated atom-photon entanglement [20,21] and violation of the CHSH inequality [22]. In the following, after presenting the general approach to the study of the detection loophole in asymmetric systems, we focus on the case where one of the systems is detected with efficiency $\eta_A = 1$ and we compute the threshold efficiency η_B^{th} for the detection of the other system. The best results are obtained for the three-setting I_{3322} inequality [23]. In analogy to Eberhard's result [14], we show that nonmaximally entangled states require a lower efficiency; moreover, here, the threshold goes down to ∼43%. Then we study two noise models: background noise and noisy detectors. Finally, we discuss the feasibility of experiments in the light of these results.

General approach.—Let us consider a typical Bell test scenario. Two distant observers, Alice and Bob, share some quantum state ρ_{AB}. Each of them chooses randomly between a set of measurements (settings) $\{A_i\}_{i=1,\dots,N_A}$ for Alice, $\{B_j\}_{j=1,\dots,N_B}$ for Bob. The result of the measurement is noted a, b. Here we will focus on dichotomic observables (corresponding to von Neumann measurements on qubits) and Alice and Bob will use the same number of settings, i.e., $a, b \in \{0, 1\}$ and $N_A = N_B \equiv N$. Repeating the experiment many times, the two parties can determine the joint probabilities $p(a, b|A_i, B_j)$ for any pairs of settings, as well as marginal probabilities $p(a|A_i)$ and $p(b|B_j)$. A Bell inequality is a constraint on those probabilities, which is satisfied for all LHV models. We say that a quantum state is nonlocal if and only if there are measurement settings such that a Bell inequality is violated. Mathematically speaking a Bell inequality is a polynomial of joint and marginal probabilities. In the case $N = 2$ the only relevant Bell inequality is the CHSH inequality, which is defined here using the Clauser-Horne polynomial [24],

$$I_{\mathrm{CHSH}} = P(A_1B_1) + P(A_1B_2) + P(A_2B_1) - P(A_2B_2) - P(A_1) - P(B_1), \quad (1)$$

where $P(A_iB_j)$ is a shortcut for $P(00|A_iB_j)$, the probability that $a = b = 0$. The bound for LHV models is $I_{\mathrm{CHSH}} \leq 0$, while quantum mechanics can reach up to $I_{\mathrm{CHSH}} = \frac{1}{\sqrt{2}} - \frac{1}{2}$. We also introduce the Bell polynomial

$$I_{3322} = P(A_1B_1) + P(A_1B_2) + P(A_1B_3) + P(A_2B_1)P(A_2B_2) + P(A_3B_1) - P(A_2B_3) - P(A_3B_2) - 2P(A_1) - P(A_2) - P(B_1), \quad (2)$$

which is the only relevant Bell inequality for the case $N = 3$ [23]. The local limit is $I_{3322} \leq 0$ and quantum mechanics violates I_{3322} up to $\frac{1}{4}$.

As an introductory example, consider the case where Alice and Bob share maximally entangled states and detect their particles with the same limited efficiency η; since they must always produce an outcome, they agree to output "0" in case of no detection. When both detectors fire, the CHSH inequality is maximally violated, i.e., $I_{\mathrm{CHSH}}^{d,d} \equiv Q = \frac{1}{\sqrt{2}} - \frac{1}{2}$. Here $I_{\mathrm{CHSH}}^{d,d}$ is the value obtained for the CHSH polynomial when both parties detect their particles. When only Alice's detector fires, $P(A_i) = \frac{1}{2}$, $P(B_j) = 1$, and $P(A_iB_j) = \frac{1}{2}$, therefore $I_{\mathrm{CHSH}}^{d,\varnothing} \equiv M_A = -\frac{1}{2}$; similarly, when only Bob's detector fires, $I_{\mathrm{CHSH}}^{\varnothing,d} \equiv M_B = -\frac{1}{2}$. When no detector fires, the LHV bound is reached, $I_{\mathrm{CHSH}}^{\varnothing,\varnothing} = 0$, since $P(A_i) = P(B_j) = P(A_iB_j) = 1$. Consequently, the whole set of data violates the CHSH inequality if and only if

$$\eta^2 Q + \eta(1 - \eta)(M_A + M_B) > 0, \quad (3)$$

yielding the well-known threshold efficiency $\eta > 82.84\%$.

In general, Alice and Bob test a Bell inequality $I \leq L$ on a state ρ_{AB} having two different detection efficiencies, η_A and η_B. In analogy to the previous example, Alice and Bob must choose the measurement settings $\{A_i, B_j\}$ and the value they output in the case of no detection, in order to maximize

$$I_{\eta_A,\eta_B} = \eta_A\eta_B Q + \eta_A(1 - \eta_B)M_A + (1 - \eta_A)\eta_B M_B + (1 - \eta_A)(1 - \eta_B)X, \quad (4)$$

where $Q = \mathrm{Tr}(\mathcal{I}\rho_{AB})$ is the mean value of the Bell operator $\mathcal{I}$ associated to the inequality, $M_{A,B}$ and X are the values of I obtained when one or both detectors do not fire. We stress that the measurement settings that maximize I_{η_A,η_B} are *not* those that maximize Q for the same quantum state, except for the maximally entangled state. Concerning the values assigned to the outputs by Alice and Bob in the case of no detection, we limit ourselves to dichotomic outcomes, i.e., $a, b \in \{0, 1\}$. Note that they could also use a third outcome for no detection [15].

Case study: $\eta_A = 1$.—The general approach above can be carried out for any specific values of the efficiencies; now we consider the limit where Alice's detector is perfect, $\eta_A = 1$. Moreover, we consider inequalities such that $L = 0$. From (4) one obtains immediately that the efficiency of Bob's detector must be above the threshold

$$\eta_B > \eta_B^{\mathrm{th}} = \frac{1}{1 - Q/M_A} \quad (5)$$

in order to close the detection loophole. For any given state, the measurement settings and Bob's output in case of no detection must be chosen as to maximize $|Q/M_A|$. Note that $M_A \leq L = 0$, because the events where only Alice's detector clicks never violate the Bell inequality.

Consider first pure states. For the *maximally entangled state*, one obtains $\eta_B^{\mathrm{th}} = \frac{1}{\sqrt{2}} \approx 70.7\%$ for the CHSH inequality and $\eta_B^{\mathrm{th}} = \frac{2}{3} \approx 66.7\%$ for the I_{3322} inequality (the settings are those that achieve $Q = \frac{1}{4}$ [23], and in the absence of detection Bob outputs 0 leading to $M_A = -\frac{1}{2}$). Note that a LHV model is known, which reproduces the correlations of the maximally entangled state under the assumption $\eta_A = 1$ and $\eta_B = 50\%$ [25]; it is an interesting open question to close this gap by finding either a better Bell-type inequality, or a better LHV model.

For *pure nonmaximally entangled states* $|\psi_\theta\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$, we performed a numerical minimization of η_B^{th}: we find that η_B^{th} decreases with decreasing θ both for CHSH and I_{3322}, as shown in Fig. 1 (thick lines), in analogy with Eberhard's result [14]. The optimal settings can always be found to lie in the (x, z) plane of the Bloch sphere, i.e., $\mathcal{A}_i = \cos(\alpha_i)\sigma_z + \sin(\alpha_i)\sigma_x$ and $\mathcal{B}_j = \cos(\beta_j)\sigma_z + \sin(\beta_j)\sigma_x$. In the case of no detection, we found that it is optimal for Bob to output always 0; note that in this case, $M_A = P(A_1) - 1 = \frac{1}{2}(\langle\psi_\theta|\mathcal{A}_1 \otimes \mathbb{1}|\psi_\theta\rangle - 1)$ for both inequalities we consider here, CHSH and I_{3322}.

In the limit of weakly entangled states ($\theta \to 0$), one finds $\eta_B^{\mathrm{th}} \to 50\%$ for CHSH and $\eta_B^{\mathrm{th}} \to \sim 43\%$ for I_{3322} (see Fig. 1). This is our main result. It is remarkable that the detection loophole can in principle be closed with $\eta_B < 50\%$. Though we could not find an analytical expression for the optimal settings as a function of θ, we provide a numerical example: for $\theta = \frac{\pi}{100}$, I_{3322} gives $\eta_B^{\mathrm{th}} \simeq 43.3\%$ ($Q \simeq 0.0013$ and $M_A \simeq -0.001$) for the optimal settings $\alpha_0 = -0.0012\pi$, $\alpha_1 = 0.1331\pi$, $\alpha_2 = 0.5494\pi$, $\beta_0 = 0.0101\pi$, $\beta_1 = -0.0038\pi$, and $\beta_2 = -0.0924\pi$.

We have seen that η_B^{th} decreases with the degree of entanglement for pure states. However, the violation of the inequality decreases as well. It is therefore important to study the effect of noise. We consider two models of noise. The first is background noise as in Ref. [14]: Alice and Bob share a state of the form

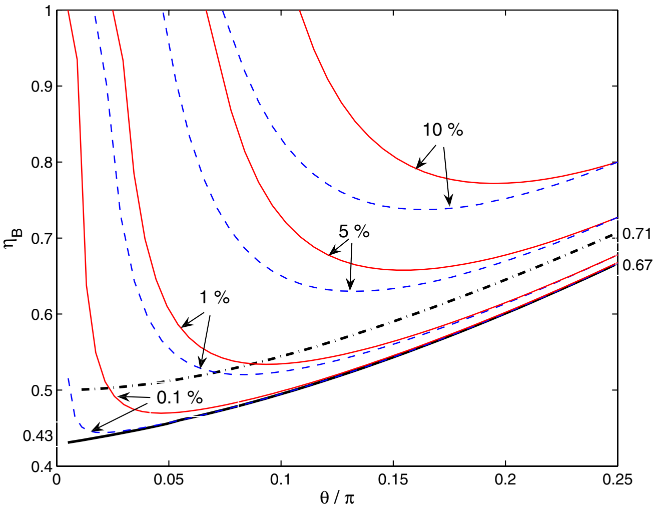

FIG. 1 (color online). Numerical optimization of the threshold efficiency η_B^{th} as a function of θ. Thick lines: results for pure states, for CHSH (dashed-dotted line) and I_{3322} (full line). For I_{3322} the threshold efficiency η_B^{th} goes down to ~43% in the limit of weakly entangled states. Thin full lines: I_{3322} with background noise, i.e., states (6); thin dashed lines: I_{3322} with detection errors, i.e., states (7); with error value for both.

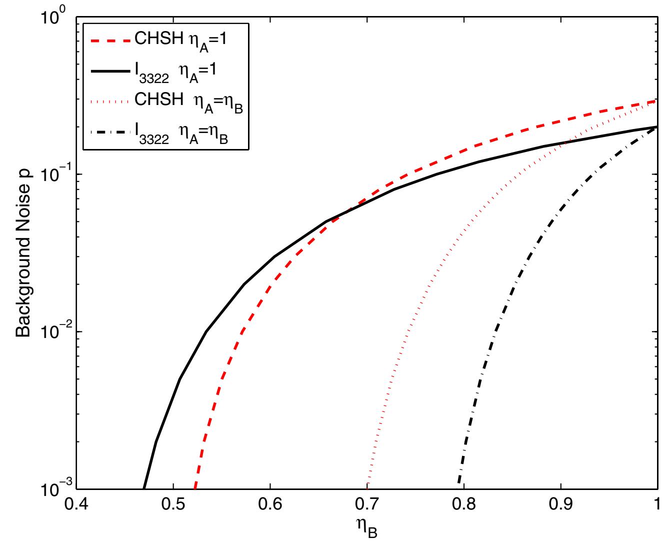

FIG. 2 (color online). Minimal detection efficiency η_B required for a given noise power. The curves for the symmetric case, $\eta_A = \eta_B$, are also plotted, the curve for CHSH being Eberhard's result. Though CHSH provides a smaller threshold efficiency for any noise power than I_{3322} in the symmetric case, I_{3322} can tolerate smaller efficiencies than CHSH when $p < 6\%$ and $\eta_A = 1$. This is probably due to the fact that inequality I_{3322} is asymmetric, contrary to CHSH.

$$\rho_{AB} = (1-p)|\psi_\theta\rangle\langle\psi_\theta| + p\frac{\mathbb{1}}{4}. \quad (6)$$

For $\theta = \frac{\pi}{4}$, the state (6) is the Werner state. The threshold efficiency for I_{3322} as a function of θ is shown in Fig. 1 (thin full lines). As expected, when θ decreases, the threshold efficiency reaches a minimum: for less entangled states the violation of the inequality is rapidly overcome by the noise. In Fig. 2, one sees that the I_{3322} inequality can tolerate lower efficiencies than the CHSH inequality for $p \lesssim 6\%$.

Another noise model, probably more relevant for experiments, supposes that Alice's and Bob's detectors have a certain probability of error, ε_d^A and ε_d^B, e.g., due to dark counts in the case of photon detection. The statistics are then described by the state

$$\rho_{AB} = (1-\varepsilon_d^A)(1-\varepsilon_d^B)|\psi_\theta\rangle\langle\psi_\theta| + \varepsilon_d^A\left(\frac{\mathbb{1}}{2}\otimes\rho_B\right) + \varepsilon_d^B\left(\rho_A\otimes\frac{\mathbb{1}}{2}\right) + \varepsilon_d^A\varepsilon_d^B\frac{\mathbb{1}}{4}, \quad (7)$$

where $\rho_A = \mathrm{Tr}_B|\psi_\theta\rangle\langle\psi_\theta|$ and $\rho_B = \mathrm{Tr}_A|\psi_\theta\rangle\langle\psi_\theta|$ are the reduced states of Alice and Bob. In the recent atom-photon experiment done in Munich [20], the atom measurement has $\eta_A \approx 1$ and $\varepsilon_d^A \approx 5\%$, whereas the photon measurement is much less efficient but also less noisy. In the light of this, we focus for definiteness on the case $\eta_A = 1$ and $\varepsilon_d^B = 0$. Again, the computed threshold efficiency as a function of θ is shown in Fig. 1 (thin dashed lines). The behavior is qualitatively the same as for the background noise, but the threshold efficiencies are lower. We have also found that I_{3322} can tolerate higher error rates than CHSH as soon as $\eta_B < 75\%$. Note that for both noise models, the optimal settings can be found to lie in the (x, z) plane of the Bloch sphere and that the optimal strategy for Bob in the case of no detection is to output always 0.

Experimental feasibility.—Atom-photon entanglement has been demonstrated both with Cd ions in an asymmetric quadrupole trap [21,22] and with Rb atoms in an optical dipole trap [20]. Nonmaximally entangled atom-photon states were already created in Ref. [21]. The overall photon detection efficiency is very low in these experiments, mostly due to inefficient photon collection. The collection efficiency could be brought to the required level by placing the atom inside a high-finesse cavity. For example, Ref. [26] demonstrated coupling of a trapped ion to a high-finesse cavity and achieved $\beta = 0.51$, where β is the fraction of spontaneously emitted photons that are emitted into the cavity mode. The experimental conditions in Ref. [27] correspond to β very close to 1. In real experiments there are other sources of loss, such as propagation losses and detector inefficiency. However, detection efficiencies of order 90% have already been achieved [28], and propagation losses can be kept small for moderate distances (see below). Overall, the perspective for closing the detection loophole for two well-separated systems seems excellent using atom-photon implementations.

Performing a loophole-free Bell experiment requires enforcing locality of the measurements [4,5] in addition to closing the detection loophole. The measurement of the atomic state, which is typically based on detecting fluorescence from a cycling transition, is relatively slow. As a consequence, enforcing locality in an experiment with atom-photon pairs requires a large separation between the two detection stations for the atom and the photon. For example, Ref. [18] estimated that for trapped Ca ions the atomic measurement could be performed in 30 μs, assuming that 2% of the photons from the cycling transition are collected, leading to a required separation of order 5 km for an asymmetric configuration [29]. For distances of this order propagation losses for the photon become significant. For example, 5 km of telecom fiber have a transmission of order 80% for the optimal wavelength range around 1.5 μm, but only of order 30% for wavelengths around 850 nm [30]. Provided that one can achieve fast atomic measurements, a photon wavelength that minimizes propagation losses, and highly efficient photon detection, a loophole-free Bell experiment might be possible with asymmetric atom-photon systems.

Conclusions and outlook.—We discussed the detection loophole in asymmetric Bell tests. In particular, we showed that, for the inequality I_{3322}, a minimal detection efficiency of $\eta_B = 43\%$ can be tolerated (for $\eta_A = 1$), considering nonmaximally entangled states. For maximally entangled states, the threshold efficiency is $\eta_B = 66.7\%$. For these states the LHV model of Ref. [25], based on the detection loophole, provides a lower bound for the threshold efficiency $\eta_B > 50\%$. It is an interesting question wether this bound can be reached by considering other Bell inequalities. We have found no improvement using the following inequalities: I_{4422} and $I_{3422}^{1,2,3}$ from Ref. [23], A_5 from Ref. [31], and $AS_{1,2}$ from Ref. [32]. From an experimental point of view, we have argued that atom-photon entanglement seems promising for closing the detection loophole for well-separated systems. We also briefly discussed the experimental requirements for realizing a loophole-free Bell experiment using an asymmetric approach.

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Note added in proof.—While finishing the writing of this manuscript, we became aware that the results presented here about the CHSH inequality were independently derived by Cabello and Larsson [33].

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