New Higher-Derivative R^4 Theorems for Graviton Scattering

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The nonminimal pure spinor formalism for the superstring is used to prove two new multiloop theorems which are related to recent higher-derivative R^4 conjectures of Green, Russo, and Vanhove. The first theorem states that when $0 < n < 12$, $\partial^n R^4$ terms in the Type II effective action do not receive perturbative contributions above $n/2$ loops. The second theorem states that when $n \leq 8$, perturbative contributions to $\partial^n R^4$ terms in the IIA and IIB effective actions coincide. As shown by Green, Russo, and Vanhove, these results suggest that $d = 4$ $N = 8$ supergravity is ultraviolet finite up to eight loops.

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Introduction.—The main success of superstring theory is that it provides a consistent quantum theory of gravity which replaces general relativity at small distances. It is therefore important to know how graviton scattering at high energies differs in superstring theory from general relativity. For scattering amplitudes involving two incoming and two outgoing gravitons, the lowest-order deviation from general relativity comes from R^4 terms in the superstring effective action [1], and higher-order deviations come from $\partial^n R^4$ terms where ∂^n denotes n spacetime derivatives, R denotes the Riemann tensor, and Lorentz indices are suppressed. The structure of these $\partial^n R^4$ terms is believed to be tightly constrained by duality symmetries of the superstring [2]; however, these duality symmetries are difficult to prove since they involve nonperturbative effects. It is therefore important to compute $\partial^n R^4$ terms in perturbative superstring theory, both for studying deviations from general relativity and for testing the nonperturbative duality symmetries.

A further motivation for studying the structure of $\partial^n R^4$ terms in Type II superstring theory is that they provide information about the maximally supersymmetric version of gravity in four dimensions which is called $d = 4$ $N = 8$ supergravity. In the early days of supersymmetry, $d = 4$ $N = 8$ supergravity was conjectured to be a consistent finite theory of gravity. It was later realized that $d = 4$ $N = 8$ supergravity probably has ultraviolet divergences which are eliminated only after including the massive states of superstring theory. However, the existence of these ultraviolet divergences in $d = 4$ $N = 8$ supergravity has never been proven, and it was recently shown by explicit computation that they are absent up to three loops [3]. Furthermore, it was argued in [2,4,5] that finiteness properties of $d = 4$ $N = 8$ supergravity are related to nonrenormalization theorems of $\partial^n R^4$ terms. For example, using the results described here that $\partial^n R^4$ terms do not get contributions above $n/2$ loops when $n < 12$, it was deduced in [5] that $d = 4$ $N = 8$ supergravity is ultraviolet finite up to 8 loops.

Although there are several prescriptions available for computing superstring amplitudes, the most efficient prescription is based on the pure spinor formalism which is manifestly super-Poincaré covariant [6]. For example, two-loop computation using the Ramond-Neveu-Schwarz (RNS) formalism is an arduous task because of the lack of manifest spacetime supersymmetry. And computations beyond one loop using the Green-Schwarz formalism are complicated by the lack of manifest Lorentz covariance. On the other hand, two-loop computations using the pure spinor formalism are relatively simple [7].

Using the minimal version of the pure spinor formalism which involves a bosonic ghost λ^α satisfying the pure spinor constraint $\lambda\gamma^m\lambda = 0$, a multiloop prescription was defined in [8] and used to prove certain vanishing theorems. One theorem stated that massless N-point multiloop amplitudes are vanishing when $N \leq 3$, which is related to perturbative finiteness of the super string [9]. Another theorem stated that R^4 terms in the Type II effective action do not receive perturbative contributions above one loop, which is related to S duality of the Type IIB superstring [10,11].

Recently, a new multiloop prescription was proposed using a nonminimal version of the pure spinor formalism which involves both the pure spinor λ^α and its complex conjugate $\bar{\lambda}_\alpha$ [12]. This nonminimal prescription for multiloop amplitudes has several advantages over the minimal prescription. First, unlike in the minimal formalism, the nonminimal formalism allows the construction of a composite b ghost satisfying $\{Q, b\} = T$. Second, the nonminimal formalism can be interpreted as a critical topological string, so the nonminimal amplitude prescription is the same as in bosonic string theory. Third, there is no need for picture-changing operators in the nonminimal prescription, which were inconvenient in the minimal prescription since they broke manifest Lorentz covariance at intermediate stages in the computation.

The only difficulty in the nonminimal prescription is regularizing the functional integral over the pure spinor ghosts when $\lambda \to 0$, which is discussed in [13]. Fortunately, this $\lambda \to 0$ regularization is unnecessary for proving the multiloop theorems in this Letter. Furthermore, it was recently shown with Carlos Mafra that various one-and two-loop amplitude computations using the nonminimal prescription correctly reproduce the RNS result [14]. Although it should be possible to directly prove the equivalence of the minimal and nonminimal amplitude prescriptions (perhaps using the Cech description of Nekrasov [15]), this has not yet been done.

In this Letter, the nonminimal prescription will be used to prove two new multiloop theorems which are related to recent higher-derivative R^4 conjectures of Green, Russo, and Vanhove based on duality symmetries [2,16–21]. The first new multiloop theorem states that when $0 < n < 12$, $\partial^n R^4$ terms in the Type II effective action do not receive genus g contributions for $g > n/2$. The restriction that $n < 12$ is related to the fact that $\partial^{12} R^4$ can be written as a superspace integral over 32θ's. The second new multiloop theorem states that when $n \leq 8$, perturbative contributions to $\partial^n R^4$ terms in the IIA and IIB effective action coincide. For $n \leq 4$, this can be shown using the RNS formalism [19], and for $n = 6$, it was recently conjectured by Green and Vanhove [17].

After reviewing the amplitude prescription using the nonminimal formalism, the two new multiloop theorems will be proven.

Review of nonminimal amplitude prescription.—Using the nonminimal pure spinor formalism, the N-point g-loop ampitude prescription is [12]

$$\mathcal{A} = \int d^{6g-6}\tau \left\langle \left| \prod_{j=1}^{3g-3} \left[\int dy_j \mu_j(y_j) b(y_j) \right] \right|^2 \times \prod_{r=1}^{N} \int d^2 z_r U_r(z_r) |\mathcal{N}|^2 \right\rangle, \quad (2.1)$$

where $\int d^{6g-6}\tau \langle | \prod_{j=1}^{3g-3} [\int dy_j \mu_j(y_j) b(y_j)] |^2 \prod_{r=1}^{N} \int d^2 z_r U_r(z_r) \rangle$ is the usual amplitude prescription of bosonic string theory,

$$b = s^\alpha \partial \bar\lambda_\alpha + \frac{\bar\lambda_\alpha [2\Pi^m (\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha]}{4(\bar\lambda\lambda)} + \frac{(\bar\lambda\gamma^{mnp} r)(d\gamma_{mnp} d + 24 N_{mn}\Pi_p)}{192(\bar\lambda\lambda)^2} - \frac{(r\gamma_{mnp} r)(\bar\lambda\gamma^m d) N^{np}}{16(\bar\lambda\lambda)^3} + \frac{(r\gamma_{mnp} r)(\bar\lambda\gamma^{pqr} r) N^{mn} N_{qr}}{128(\bar\lambda\lambda)^4} \quad (2.2)$$

is a composite operator satisfying $\{Q, b\} = T$, and

$$\mathcal{N} = \exp[-\lambda^\alpha \bar\lambda_\alpha - r_\alpha \theta^\alpha - \tfrac{1}{2} N^I_{mn} \bar N^{mnI} - J^I \bar J^I - \tfrac{1}{4}(s^I \gamma^{mn} \bar\lambda)(d^I \gamma_{mn}\lambda) - (\bar\lambda s^I)(\lambda d^I)] \quad (2.3)$$

is an operator satisfying $\mathcal{N} = e^{\{Q,\chi\}}$ which regularizes the $0/0$ coming from integration over the world sheet zero modes where, for example, $s^{\alpha I} = \oint_{a_I} dz s^\alpha$ is the zero mode obtained by integrating s^α around the Ith a cycle.

When the bosonic zero modes of λ^α, N^I_{mn}, or J^I go to infinity, the functional integral over these zero modes is well defined because of the exponential cutoff in $\mathcal{N}$. However, when $\lambda^\alpha \to 0$, the poles in (2.2) make the functional integral over λ^α ill defined if the sum of the degree of the poles is greater than or equal to 11. If the contributions from the b ghosts diverge as fast as $(\lambda\bar\lambda)^{-11}$, the measure factor $\int d^{11}\lambda d^{11}\bar\lambda$ does not converge fast enough to make the functional integral well defined. In a paper with Nekrasov [13], it will be shown how to regularize this $\lambda \to 0$ divergence for arbitrary multiloop amplitudes. However, there are certain amplitudes for which the sum of the degree of the poles from the b ghosts is always less than 11, so one does not need to worry about regularizing the $\lambda \to 0$ divergence. For example, since the maximum pole in the b ghost is of degree 3, there is no $\lambda \to 0$ divergence when the genus is less than or equal to two since, for these amplitudes, there are three or fewer b ghosts. As will also be shown in [13], another type of amplitude for which there is no $\lambda \to 0$ divergence is when at least one of the 16 θ^α zero modes comes from $\mathcal{N}$.

Since the b ghost of (2.2) is spacetime supersymmetric, the 16 θ^α zero modes in the functional integral of (2.1) must come either from the regulator $\mathcal{N}$ or from the external vertex operators $\prod_{r=1}^N U_r(z_r)$. If all 16 θ zero modes come from the superfields in the vertex operators U_r, the resulting term in the effective action is not a ten-dimensional F term since it can be written as an integral over the maximum number of θ's. However, if at least one of the θ zero modes come from $\mathcal{N}$, the amplitude could contribute to F terms. Therefore, the above argument implies that amplitudes which contribute to ten-dimensional F terms do not require regularization when $\lambda \to 0$.

Note that as in lower dimensions, $D = 10$ F terms are defined as manifestly gauge-invariant terms in the superspace effective action which cannot be written as integrals over the maximum number of θ's. Although one does not know how to construct off-shell $D = 10$ superspace actions, one can construct higher-derivative $D = 10$ superspace actions which are functions of on-shell linearized superfields.

For closed Type IIB superstrings, the massless supergravity vertex operator is

$$\int d^2 z U = \int dz ([G_{MN}(Z) + B_{MN}(Z)] \partial Z^M \bar\partial Z^N + W^{\alpha\beta}(Z) d_\alpha \bar d_\beta + \ldots), \quad (2.4)$$

where $Z^M = (x^m, \theta^\alpha, \bar\theta^\beta)$, and the gauge-invariant superfield of lowest dimension is $W^{\alpha\beta}(x, \theta, \bar\theta)$ whose lowest component is the Ramond-Ramond field strength of dimension 1. Note that the dilaton and axion are gauge-invariant fields of dimension zero, but they always appear with derivatives in the massless vertex operator. Since $N = 2$ $D = 10$ superspace contains 32θ's, any term in the superspace action involving M superfields $W^{\alpha\beta}$ which is integrated over the full superspace has dimension $\geq (M + 16)$. Therefore, any term in the $N = 2$ $D = 10$ superspace action involving M field strengths which has dimension less than $(M + 16)$ is necessarily an $N = 2$ $D = 10$ F term. For example, since the curvature tensor R_{mnpq} has dimension 2, the term $\int d^{10}x\sqrt{g}\partial^L R^M$ in the Type II effective action is an $N = 2$ $D = 10$ F term if $L + 2M < M + 16$, i.e., if $L + M < 16$.

New multiloop theorems.—The multiloop theorems in this Letter will be proven by counting fermionic zero modes in the integrand of (2.1). The left-moving fermionic world sheet fields in the nonminimal formalism include the superspace variable θ^α and its conjugate momentum d_α and the nonminimal variable r_α and its conjugate momentum s^α. The nonminimal variable r_α is constrained to satisfy $\bar\lambda\gamma^m r = 0$, where $\bar\lambda\gamma^m\bar\lambda = 0$, so r_α has 11 independent components. Since θ^α and r_α are world sheet scalars, on a genus g surface θ^α contains 16 zero modes, d_α contains $16g$ zero modes, r_α contains 11 zero modes, and s^α contains $11g$ zero modes. So the amplitude of (2.1) vanishes unless all of these fermionic zero modes are present in the integrand of (2.1).

Nonrenormalization of $\partial^n R^4$.—Using the prescription of (2.1), it will now be proven that perturbative contributions to $\partial^n R^4$ terms vanish above $n/2$ loops. This is proven by showing that the massless four-point g-loop amplitude at low energies is proportional to

$$\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}\right)^{2g+4} W^4 = \partial^{2g}R^4 + \ldots, \qquad (3.1)$$

where W is the Ramond-Ramond field strength of (2.4). So $\partial^n R^4$ terms only get perturbative contributions up to genus $n/2$. When $g \geq 6$, (3.1) is no longer an F term, so there may be $\lambda \to 0$ divergences which need to be regularized. The theorem has therefore only been proven when $n < 12$.

To get a nonvanishing four-point g-loop amplitude, the integrand of (2.1) must provide $16g$ d_α zero modes which can come either from the four vertex operators of (2.4), from the regulator $\mathcal{N}$ of (2.3), or from the $3g - 3$ b ghosts of (2.2). The most efficient way to obtain these $16g$ zero modes is if the four vertex operators provide the term $(W^{\alpha\beta}d_\alpha\bar d_\beta)^4$ and the regulator $\mathcal{N}$ provides the term $(sd)^{11g}$, where $11g$ is the maximum power since there are only $11g$ independent s zero modes.

The remaining $5g - 4$ d_α zero modes must come from the $3g - 3$ b ghosts, and to minimize the number of θ^α zero modes coming from the vertex operators, it will be advantageous to minimize the number of r_α zero modes coming from the b ghosts. The ghost contribution which provides $5g - 4$ d_α zero modes while minimizing the number of r_α zero modes is

$$\left(\frac{\partial x^m(\bar\lambda\gamma_m d)}{2(\bar\lambda\lambda)}\right)^{g-2}\left(\frac{\bar\lambda\gamma^{mnp}r(d\gamma_{mnp}d)}{192(\bar\lambda\lambda)^2}\right)^{2g-1}, \qquad (3.2)$$

where $g - 2$ b ghosts provide the first term and $2g - 1$ b ghosts provide the second term.

The contribution of (3.2) provides $2g - 1$ of the $11 r_\alpha$ zero modes, so the remaining $12 - 2g$ r_α zero modes must come from $\mathcal{N}$ through the term $(r\theta)^{12-2g}$. Since this term provides $12 - 2g$ of the $16\theta^\alpha$ zero modes, the remaining $2g + 4$ θ^α zero modes must come from the superfields $W^{\alpha\beta}$ in the vertex operators.

So for the amplitude to be nonvanishing, the four external vertex operators must provide at least $2g + 4$ θ^α zero modes. Therefore, at low energies, the four-point g-loop scattering amplitude is proportional to (3.1) as claimed. Note that it is assumed that there are no inverse factors of momentum coming from the $\langle\prod_{r=1}^4 e^{ik_r x(z_r)}\rangle$ correlation function which would decrease the number of derivatives on R^4 in (3.1). This assumption is reasonable since the massless three-point multiloop amplitude vanishes, so one does not expect any poles in momentum for the four-point multiloop amplitude.

Equivalence of IIA and IIB $\partial^n R^4$ terms.—It will now be proven that up to four loops, four-point graviton contributions to F terms coincide in the IIA and IIB effective actions. Using the previous theorem that $\partial^n R^4$ terms do not get perturbative contributions above $n/2$ loops, this implies that perturbative contributions to $\partial^n R^4$ terms coincide in the IIA and IIB effective actions for $n \leq 8$.

To prove this multiloop theorem, similar methods to [19] will be used. IIA and IIB superstrings are related by a parity operation on the left-moving world sheet variables which flips the chirality of the left-moving spacetime spinor. For graviton scattering amplitudes, this parity operation flips the sign of terms which involve an $\epsilon_{m_1\ldots m_{10}}$ tensor coming from the integration over the left-moving variables.

For four-point graviton amplitudes, the only way to contract the vector indices on such an $\epsilon_{m_1\ldots m_{10}}$ tensor is if there is also an $\epsilon_{n_1\ldots n_{10}}$ tensor coming from the integration over the right-moving variables. One can then contract the vector indices of the left-moving ϵ tensor either with the indices of the right-moving ϵ tensor or with the external momenta and polarizations.

Since there are three independent momenta, k_r^m for $r = 1$ to 3, and four independent polarizations, h_r^{mn} for $r = 1$ to 4, the minimum number of indices which must be contracted between the left- and right-moving ϵ tensors is three. This can be accomplished using the contraction

$$h_1^{m_1n_1}h_2^{m_2n_2}h_3^{m_3n_3}h_4^{m_4n_4}k_1^{m_5}k_1^{n_5}k_2^{m_6}k_2^{n_6}k_3^{m_7}k_3^{n_7}\eta^{m_8n_8}\eta^{m_9n_9}\eta^{m_{10}n_{10}}\epsilon_{m_1\ldots m_{10}}\epsilon_{n_1\ldots n_{10}}. \qquad (3.3)$$

To contract indices of the left- and right-moving ϵ tensors, the correlation function must involve factors of ∂x^m and $\bar{\partial} x^n$ since these are the only left- and right-moving fields which can be contracted. For example, in the RNS formalism, these factors of ∂x^m and $\bar{\partial} x^n$ come from the left- and right-moving picture-changing operators. Since g-loop RNS amplitudes involve $2g - 2$ left- and right-moving picture-changing operators, and since one needs at least three ∂x's and ∂x's to perform the contraction of (3.3), the term in (3.3) is only possible when $g \geq 3$. So the four-point graviton amplitudes in IIA and IIB superstring theory have been proven to coincide up to two loops using the RNS formalism [19].

Using the prescription of section for F-term computations, the three factors of ∂x^m and $\bar{\partial} x^n$ can come from the term $\frac{\partial x^m (\bar{\lambda} \gamma_m d)}{2(\bar{\lambda} \lambda)}$ in the b ghost. By counting d_α zero modes as in (3.2), one finds at genus g that the maximum number of ∂x factors is $g - 2$. So the contraction of (3.3) is only possible when $g \geq 5$, implying that four-point graviton amplitudes contribute equally to IIA and IIB F terms when $g \leq 4$.

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