Choreographic Solution to the General-Relativistic Three-Body Problem

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We reexamine the three-body problem in the framework of general relativity. The Newtonian N-body problem admits choreographic solutions, where a solution is called choreographic if every massive particle moves periodically in a single closed orbit. One is a stable figure-eight orbit for a three-body system, which was found first by Moore (1993) and rediscovered with its existence proof by Chenciner and Montgomery (2000). In general relativity, however, the periastron shift prohibits a binary system from orbiting in a single closed curve. Therefore, it is unclear whether general-relativistic effects admit choreography such as the figure eight. We examine general-relativistic corrections to initial conditions so that an orbit for a three-body system can be choreographic and a figure eight. This illustration suggests that the general-relativistic N-body problem also may admit a certain class of choreographic solutions.

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Introduction.—The three-body problem in the Newton gravity is one of the classical problems in astronomy and physics (see, e.g., [1]). In 1765, Euler found a collinear solution, and Lagrange found an equilateral triangle solution in 1772. It is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. In fact, Poincarè proved that we cannot analytically obtain all of the solutions, and the number of new solutions is increasing [2]. Therefore, the three-body problem remains unsettled even for the Newtonian gravity. The Newtonian N-body problem admits choreographic solutions, which have attracted increasing interest. Here, a solution is called choreographic in the celestial mechanics if every massive particle moves periodically in a single closed orbit. In fact, a choreographic figure-eight solution to the three-body problem was found first by Moore [3] and rediscovered with its existence proof by Chenciner and Montgomery [4].

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations. Especially, it has passed "classical" tests, such as the deflection of light, the perihelion shift of Mercury and the Shapiro time delay, and also a systematic test using the remarkable binary pulsar "PSR 1913+16" [5]. It is worthwhile to examine the three-body (or, more generally, N-body) problem in general relativity. N-body dynamics in the general-relativistic (GR) gravity plays important roles in astrophysics. For instance, the formation of massive black holes in star clusters is tackled mostly by Newtonian N-body simulations (see, e.g., [6]). However, it is difficult to work out in general relativity compared with the Newtonian gravity, because the Einstein equation is much more complicated [7] (even for a two-body system [8–11]). In addition, future space astrometric missions such as SIM and GAIA [12–14] require a general-relativistic modeling of the solar system within the accuracy of a microarcsecond [15]. Furthermore, a binary plus the third body were discussed also for perturbations of gravitational waves induced by the third body [16–19]. In this Letter, we do not intend to solve the N-body problem in general relativity under a general situation. Instead, we shall focus on a choreographic solution. No choreographic solution has been found to the general-relativistic N-body problem so far.

In a two-body system, the post-Newtonian corrections cause the periastron shift so that the binary system cannot orbit in a single closed curve as shown in Fig. 1 [7]. As a result, it is unclear whether general-relativistic perturbations admit a choreographic solution as the figure eight. One may thus ask, What happens for the figure-eight in Einstein's gravity? Specific questions may arise, such as: Does the figure-eight cause periastron shift? Does the figure-eight make a transition to an open orbit in the general-relativistic gravity? The purpose of this Letter is to answer these questions by carefully examining general-relativistic effects to initial conditions for being a choreographic solution.

This Letter is organized as follows. First, we briefly summarize the choreographic figure-eight solution in the Newton gravity. Next, we analytically examine initial conditions and numerically solve the Einstein-Infeld-Hoffman (EIH) equation of motion in order to obtain a choreographic solution in general relativity. Throughout this Letter, we take the units of $G = c = 1$.

Newtonian choreographic solution.—As mentioned above, it is impossible to describe all the solutions to the three-body problem even for the $1/r$ potential. The simplest periodic solutions for this problem were discovered by Euler (1765) and by Lagrange (1772). Euler's solution is a collinear solution, in which the masses are collinear at every instant with the same ratios of their distances. Lagrange's solution is an equilateral triangle solution in which each mass moves in an ellipse in such a way that the triangle formed by the three bodies revolves. Built out of Keplerian ellipses, they are the only explicit solutions. In these solutions, each mass moves on an ellipse. A choreo-

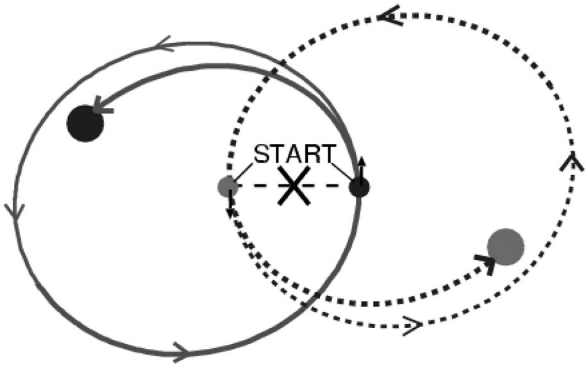

FIG. 1. A schematic figure for a binary orbit in general relativity. The orbit is not closed any more, because of the periastron shift.

graphic solution for which three bodies move periodically in a single figure-eight orbit was found first by Moore by numerical computations [3]. The existence of such a figure-eight orbit was proven by Chenciner and Montgomery [4]. This solution is stable in the Newtonian gravity [20,21]. The figure-eight orbit seems unique up to scaling and rotation according to all numerical investigations, and at the end its unicity has been recently proven [22]. Furthermore, it is shown numerically that fourth, sixth, or eighth order polynomials cannot express the figure-eight solution [21]. Nevertheless, no analytic expression in closed forms for the figure-eight trajectory has been found up to now. Therefore, in this Letter, we numerically prepare the figure-eight orbit.

For simplicity, we assume a three-body system with each mass equal to m. Without loss of the generality, the orbital plane is taken as the x-y plane. The position of each mass (m_A) is denoted by (x_A, y_A) for $A = 1, 2, 3$. Figure 2 shows the figure-eight orbit, where we take the initial condition as $\ell \equiv (x_1, y_1) = (-x_2, -y_2) = (97.00, -24.31)$, $(x_3, y_3) = (0, 0)$, and $\boldsymbol{V}_{\rm Newton} \equiv (\dot{x}_3, \dot{y}_3) = (-2\dot{x}_1, -2\dot{y}_1) = (-2\dot{x}_2, -2\dot{y}_2) = (-0.09324, -0.08647)$, where a dot denotes the time derivative [21]. When one mass arrives at the knot (center) of the figure eight, $\ell \equiv |\ell|$ is half of the separation between the remaining two masses. It is convenient to use ℓ instead of a distance between the knot and the apoapsis, because the inertial moment is expressed simply as $2m\ell^2$. The orbital period is estimated as $T_{\rm Newton} = 6.326 m^{-1/2}\ell^{3/2} \approx 10^4 (M_\odot/m)^{1/2}(\ell/R_\odot)^{3/2}$ sec, where $M_\odot$ and $R_\odot$ are the solar mass and radius, respectively. Obviously this system has no Killing vector as seen in Fig. 2. Here, we should note that ℓ is taken as 100, while it is unity in previous works. This is because we will treat the post-Newtonian correction in terms of the ratio between the mass and the separation such as ℓ. In our case, the ratio m/ℓ is 0.01; that is, the post-Newtonian correction becomes about 1%. In the equation of motion, the second post-Newtonian (2PN) corrections of the order of $(m/\ell)^3$ can be safely neglected, if m/ℓ is very small, say, 10^{-8}. In this case, however, the Newtonian and relativistic orbits will be indistinguishable. In order to demonstrate the difference between the two orbits, we choose m/ℓ as 0.01, for which the first post-Newtonian (1PN) terms are several dozens times larger than 2PN terms. In our computations, which are not long-time integrations over, say, thousands orbital periods, we can assume that 1PN terms are enough to bring major relativistic effects.

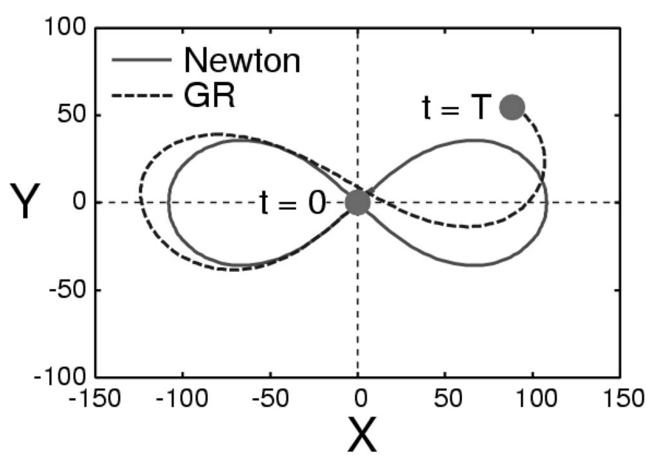

FIG. 2. Figure-eight orbits starting at the Newtonian initial condition. The solid curve denotes a figure-eight orbit in the Newtonian gravity. The dashed curve denotes a trajectory of one mass following the EIH equation of motion under the same Newtonian initial condition.

Post-Newtonian figure-eight orbit.—In the previous part, the motion of massive bodies follows the Newtonian equation of motion. In order to include the dominant part of general-relativistic effects, we take account of the terms at the first post-Newtonian order. Namely, the motion of the massive bodies obeys the EIH equation of motion [7]. The EIH equation is derived also from the first post-Newtonian Lagrangian as [23]

$$\mathcal{L} = \frac{1}{2}\sum_A m_A v_A^2 + \frac{1}{2}\sum_A \sum_{B\neq A} \frac{m_A m_B}{r_{AB}} + \frac{1}{8}\sum_A m_A v_A^4 - \frac{1}{4}\sum_A \sum_{B\neq A} \frac{m_A m_B}{r_{AB}}[7(\boldsymbol{v}_A \cdot \boldsymbol{v}_B) - 6v_A^2 + (\boldsymbol{v}_A \cdot \boldsymbol{n}_{AB})(\boldsymbol{v}_B \cdot \boldsymbol{n}_{AB})] - \frac{1}{2}\sum_A \sum_{B\neq A} \sum_{C\neq A} \frac{m_A m_B m_C}{r_{AB} r_{AC}}, \tag{1}$$

where we define

$$\boldsymbol{r}_{AB} \equiv \boldsymbol{r}_A - \boldsymbol{r}_B, \tag{2}$$

$$r_{AB} \equiv |\boldsymbol{r}_{AB}|, \tag{3}$$

$$\boldsymbol{n}_{AB} \equiv \frac{\boldsymbol{r}_{AB}}{r_{AB}}. \tag{4}$$

Figure 2 shows an orbit of a body starting at the Newtonian initial condition described above. In Fig. 2, a figure-eight orbit does not seem to survive at the 1PN order.

However, this is not the case. We should note that the initial condition at the 1PN order does not necessarily coincide with that for the Newtonian gravity. We will thus carefully examine the initial condition by taking account of 1PN corrections. For this purpose, we assume that both the linear and the angular momenta are zero (i.e., $\boldsymbol{P} = \boldsymbol{0}$ and $\boldsymbol{L} = \boldsymbol{0}$).

We should remember $\boldsymbol{v}_1 = \boldsymbol{v}_2 = -(\boldsymbol{v}_3)/2$ for the Newtonian figure-eight orbit, for which both the total linear and angular momenta are zero. The changes in $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are expressed by using two vectors $\boldsymbol{v}_3$ and ℓ, which are linearly independent.

Hence, the initial velocity of each mass is parametrized as

$$\boldsymbol{v}_1 = k\boldsymbol{V} + \xi \frac{m}{\ell^3}(\boldsymbol{V}\cdot\ell)\ell, \tag{5}$$

$$\boldsymbol{v}_2 = k\boldsymbol{V} + \xi \frac{m}{\ell^3}(\boldsymbol{V}\cdot\ell)\ell, \tag{6}$$

$$\boldsymbol{v}_3 = \boldsymbol{V}, \tag{7}$$

where k is expressed as

$$k = -\frac{1}{2} + \alpha|\boldsymbol{V}|^2 + \beta\frac{m}{\ell}. \tag{8}$$

Here, the 1PN terms have either $|\boldsymbol{V}|^2$ or m/ℓ. If $\boldsymbol{P} = \boldsymbol{0}$ and $\boldsymbol{L} = \boldsymbol{0}$, there is no need of $|\boldsymbol{V}|^2$ in front of ℓ in Eqs. (5) and (6), as shown below.

The linear and angular momenta are calculated from the first post-Newtonian Lagrangian [23]. Here, we impose the condition of $\boldsymbol{P} = \boldsymbol{0}$ and $\boldsymbol{L} = \boldsymbol{0}$ at 1PN order. Then, we determine the 1PN coefficients as

$$\alpha = -\tfrac{3}{16}, \tag{9}$$

$$\beta = \tfrac{1}{8}, \tag{10}$$

$$\xi = \tfrac{1}{8}. \tag{11}$$

Up to this point, $\boldsymbol{V}$ is arbitrary. Next, we determine $\boldsymbol{V}$.

The initial velocity of the particles can be different from that for the Newtonian gravity. The post-Newtonian effects affect both the magnitude and the direction of the velocity. Therefore, by using two linearly independent vectors, ℓ and $\boldsymbol{V}_{\text{Newton}}$, we parametrize the initial velocity as

$$\boldsymbol{V} = \left(1 + \delta\frac{m}{\ell}\right)\boldsymbol{V}_{\text{Newton}} + \eta\frac{m}{\ell}\frac{\ell}{\ell}\left(\boldsymbol{V}_{\text{Newton}}\cdot\frac{\ell}{\ell}\right), \tag{12}$$

where it is sufficient to express 1PN corrections in terms of either m/ℓ or $|\boldsymbol{V}_{\text{Newton}}|^2$ in numerical computations, though both are necessary for analytic calculations of $\boldsymbol{P} = \boldsymbol{0}$ and $\boldsymbol{L} = \boldsymbol{0}$. For convenience sake, we choose m/ℓ in Eq. (12).

By numerically performing trial and error iterations until achieving a periodic orbit, we find

$$\delta = -3.3, \tag{13}$$

$$\eta = -3.7. \tag{14}$$

Here, the iterative computations are done until we find the values of δ and η for which the three masses simultaneously return to their initial positions. Our procedure is as follows. If and only if one particle returns to the neighborhood of its initial position (i.e., the origin for the particle labeled by 3) within the positional deviation of 0.01, we measure how far the remaining two particles are from their initial positions at the same moment when the particle is closest to the initial position. For Eqs. (13) and (14), the sum of the square distances is minimized as approximately 0.1. This is sufficient for $\ell = 100$ and $m/\ell = 0.01$, because expected positional shifts after one cycle are of the order of unity or more. For instance, such a shift exceeds 10 for $\alpha = \beta = \xi = \delta = \eta = 0$ in Fig. 2.

We integrate the motion over 10 cycles to confirm the periodicity. After 10 periods, the found solution comes to the same point within the deviation of ± 1. The numerical computation gives the orbital period as

$$T_{\text{GR}} \approx \left(1 + \frac{6m}{\ell}\right)T_{\text{Newton}}. \tag{15}$$

The relativistic figure-eight orbit appears to be stable, in the sense that we recognize "figure-eight-like" orbits that resemble figure eight for slightly different values of δ and η. It is a future subject to analyze the long-time stability of the relativistic figure eight.

Figure 3 shows that a figure-eight orbit is still closed even after including the dominant general-relativistic effects. In Fig. 3, we can recognize an asymmetric difference between the Newtonian figure-eight orbit and the general-relativistic one. The deviation is partly fiducial, because the

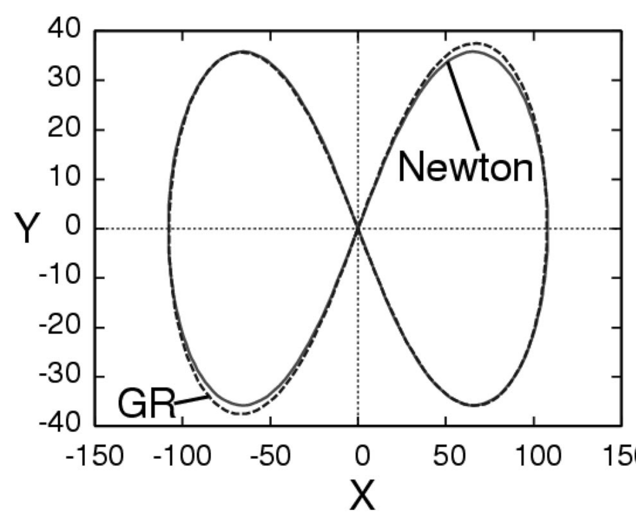

FIG. 3. Figure-eight orbits. The solid curve denotes a figure-eight orbit in the Newtonian gravity. The dashed curve denotes a figure-eight orbit at the 1PN order of general relativity.

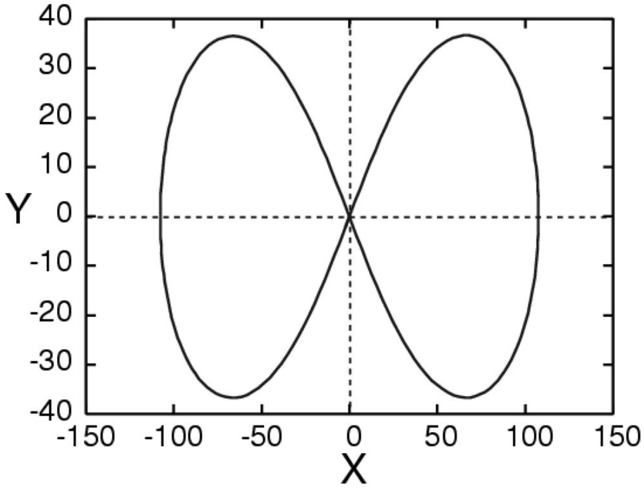

FIG. 4. Relativistic figure-eight orbit. The principal axes of the orbit are chosen as x and y axes.

principal axes of the GR figure-eight orbit are not along the x and y axes. That is, ℓ, which defines the direction of the initial position of a particle with respect to the principal axes, changes slightly at the 1PN order. The axes are inclined by 0.012 rad with respect to those of the Newtonian figure-eight orbit. Now, we choose the x axis as the principal axis for both the Newtonian figure-eight orbit and the general-relativistic one. After choosing the principal axes, Fig. 4 shows a general-relativistic choreographic solution at the first post-Newtonian order. The solution recovers line symmetry with respect to the x and y axes. There are no significant differences in the velocity between the Newtonian and GR figure-eight orbits. One may notice that the 2PN terms are neglected. It would be safer to choose m/ℓ as 10^{-8}, for instance, and then on the figures to exaggerate the differences between the Newtonian and post-Newtonian solutions in order to make them visible.

Finally, we mention the possibility of three-body systems in a choreographic orbit such as a figure eight. As a new outcome of binary-binary scattering, the figure-eight orbit was discussed for presenting a way of detecting such an orbit in numerical computations [24]. According to the numerical result, the probability of the formation of figure-eight orbits is a tiny fraction of 1%. The gravitational waves emitted by the figure-eight orbit have been recently studied by assuming the motion in the Newton gravity [25]. By evaluating the radiation reaction time scale, it is shown also that figure-eight orbit sources emitting gravitational waves may be too rare to detect.

Conclusion.—We obtained a general-relativistic initial condition for being a figure-eight orbit. This condition provides the first choreographic solution taking account of the post-Newtonian corrections. It is interesting to include higher post-Newtonian corrections, especially 2.5PN effects, in order to elucidate the backreaction on the evolution of the orbit due to the gravitational waves emission at the 2.5PN order. If the system is secularly stable against the gravitational radiation, it is probable that one might see a shrinking ($\dot{\ell} < 0$) figure-eight orbit as a consequence of a decrease in the total energy ($\dot{E} < 0$). This speculation will be confirmed or rejected in the future. It may be important also to look for other relativistic choreographic solutions for a system including four or more masses. It is possible that some of Newtonian choreographic solutions are prohibited by general-relativistic effects. Further investigations along these lines will allow us to probe many-body dynamics in the Einstein gravity.

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