Spectral Convexity for Attractive SU(2N) Fermions

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We prove a general theorem on spectral convexity with respect to particle number for 2N degenerate components of fermions. The number of spatial dimensions is arbitrary, and the system may be uniform or constrained by an external potential. We assume only that the interactions are governed by an SU(2N)-invariant two-body potential whose Fourier transform is negative definite. The convexity result implies that the ground state is in a 2N-particle clustering phase. We discuss implications for light nuclei as well as asymmetric nuclear matter in neutron stars.

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Interacting fermions with more than two components exhibit a variety of low temperature phenomena. Of particular interest are phenomena which appear in different quantum systems and therefore could be characterized as universal. One example in three dimensions is the Efimov effect, which predicts a geometric sequence of trimer bound states for interactions in the limit of zero range and infinite scattering length [1]. Once the binding energy of the trimer is fixed it turns out that the binding energy of the four-body system is also determined [2]. In two dimensions a different geometric sequence has been predicted for the binding energy of N-body clusters in the large N limit [3].

Several recent studies have investigated pairing and the superfluid properties of three-component fermions [4]. Systems involving four-component fermions are of direct relevance to the low-energy effective theory of protons and neutrons. Because of antisymmetry there are only two S-wave nucleon scattering lengths. Some general properties of this low-energy effective theory have been studied such as pairing, the fermion sign problem, and spectral inequalities [5,6]. Wu *et al.* [7] have pointed out that the effective theory has an accidental SO(5) or Sp(4) symmetry, and several different phases such as quintet Cooper pairing or four-fermion quartetting could be experimentally realized for different scattering lengths with ultracold atoms in optical traps or lattices [8]. When the scattering lengths are equal the symmetry is expanded to an SU(4) symmetry first studied by Wigner [9].

In the following we prove a general theorem on spectral convexity with respect to particle number for 2N degenerate components of fermions. The theorem holds for any number of spatial dimensions, and the system may be either uniform or constrained by an external potential. We assume only that the interactions are governed by an SU(2N)-invariant two-body potential whose Fourier transform is negative definite. The main result is that if the ground state energy E is plotted as a function of the number of particles A, then the function $E(A)$ is convex for even A modulo 2N. Furthermore $E(A)$ for odd A is bounded below by the average of the two neighboring even values, $E(A-1)$ and $E(A+1)$. This is illustrated in Fig. 1 for both the weak attractive and strong attractive cases. This convexity pattern could be regarded as an SU(2N) generalization of even-odd staggering for the ground state energy in the attractive two-component system. We should clarify that the state labeled as $A = 2NK$ has exactly K particles of each component, while the state with $A = 2N(K+1)$ has exactly $K+1$ particles of each component. The states shown with A in between these two have $K+1$ particles for some components and K particles for the others.

A weaker form of this inequality was proven for $A \leq 2N$ and zero-range attractive interactions [6]. Here we extend the proof to any A and any SU(2N)-invariant potential with a negative-definite Fourier transform. To get a feeling for our main result it is helpful to consider a simpler system consisting of A particles with two-body forces in

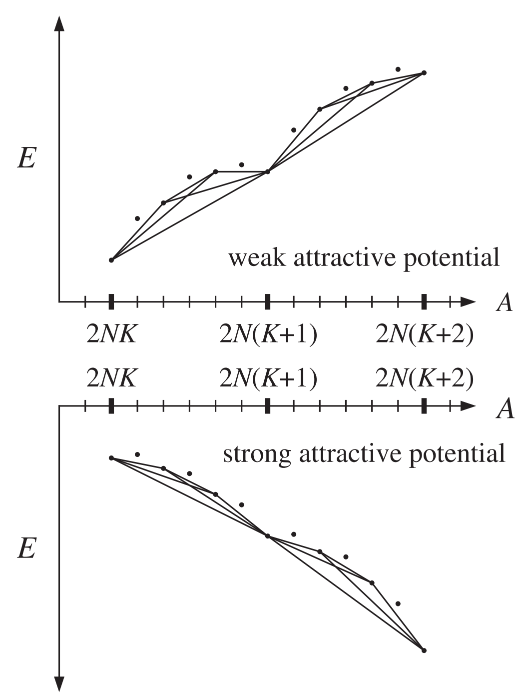

FIG. 1. Illustration of the convexity constraints for the ground state energy E as a function of particle number A. The line segments show the convexity lower bounds.

an external potential. Qualitatively the kinetic energy of this system and the energy of interaction with the external potential are each proportional to A, whereas the energy of the two-body interaction is proportional to the number of pairs, $A(A-1)/2$. Therefore if the two-body forces are purely attractive, the second derivative of the total energy d^2E/dA^2 is negative. Our main result is a rigorous theorem that establishes a relation of this sort for a system of fermions interacting with purely attractive SU($2N$)-invariant two-body forces.

We start by considering $2N$ degenerate components of nonrelativistic fermions in d spatial dimensions. We assume the interactions are governed by an SU($2N$)-invariant two-body potential $V(\vec{r})$ whose Fourier transform $\tilde{V}(\vec{p})$ is strictly negative. We also allow an SU($2N$)-invariant external potential $U(\vec{r})$ whose properties are not restricted. The general form of the Hamiltonian is

$$H = -\frac{1}{2m}\sum_{i=1,\ldots,2N}\int d^d\vec{r}a_i^\dagger(\vec{r})\vec{\nabla}^2 a_i(\vec{r}) + \int d^d\vec{r}U(\vec{r})\rho(\vec{r}) + \frac{1}{2}\int d^d\vec{r}d^d\vec{r}' : \rho(\vec{r})V(\vec{r}-\vec{r}')\rho(\vec{r}') :, \tag{1}$$

where $\rho(\vec{r})$ is the SU($2N$)-invariant density,

$$\rho(\vec{r}) = \sum_{i=1,\ldots,2N} a_i^\dagger(\vec{r})a_i(\vec{r}). \tag{2}$$

The : symbols denote the normal ordering of creation and annihilation operators.

We consider the system on a hypercubic lattice using a transfer matrix formalism. The term "transfer matrix" refers to a lattice approximation to the exponential $e^{-H\Delta t}$, where Δt equals one lattice time step. It turns out that the lattice method we discuss is an efficient computational scheme for calculating ground state properties of attractive fermionic systems. In many cases the method partially or even completely eliminates the fermion sign problem. Further details of the lattice formalism and examples of Monte Carlo calculations can be found in [10,11].

We let $\vec{n} = (\vec{n}_s, n_t)$ represent $(d+1)$-dimensional lattice vectors. The subscript s on $\vec{n}_s$ denotes a d-dimensional spatial lattice vector. We write the d-dimensional spatial lattice unit vectors as $\hat{1}, \ldots, \hat{d}$. Throughout our discussion of the lattice system we use dimensionless parameters and operators which correspond with physical values multiplied by the appropriate power of the spatial lattice spacing a. We let a_t be the temporal lattice spacing and α_t be the ratio a_t/a. L denotes the spatial length of the periodic hypercubic lattice.

We use the notation $\tilde{V}(2\pi\vec{k}_s/L)$ for the Fourier transform of the lattice potential $V(\vec{n}_s)$,

$$\tilde{V}(2\pi\vec{k}_s/L) = \sum_{\vec{n}_s} V(\vec{n}_s)e^{i2\pi\vec{n}_s\cdot\vec{k}_s/L}. \tag{3}$$

By assumption $\tilde{V}(2\pi\vec{k}_s/L)$ is strictly negative. Let M be the normal-ordered transfer matrix operator

$$M =: \exp\left[-\alpha_t H_{\text{free}} - \alpha_t\sum_{\vec{n}_s} U(\vec{n}_s)\rho(\vec{n}_s) - \frac{\alpha_t}{2}\sum_{\vec{n}_s,\vec{n}_s'}\rho(\vec{n}_s)V(\vec{n}_s-\vec{n}_s')\rho(\vec{n}_s')\right]:, \tag{4}$$

where H_{free} is the free lattice Hamiltonian,

$$H_{\text{free}} = -\frac{1}{2m}\sum_{\vec{n}_s}\sum_{\hat{l}_s=\hat{1},\ldots,\hat{d}}\sum_{i=1,\ldots,2N}\{a_i^\dagger(\vec{n}_s)[a_i(\vec{n}_s+\hat{l}_s) + a_i(\vec{n}_s-\hat{l}_s) - 2a_i(\vec{n}_s)]\}. \tag{5}$$

Let $V^{-1}(\vec{n}_s)$ be the inverse of $V(\vec{n}_s)$,

$$V^{-1}(\vec{n}_s) = \frac{1}{L^d}\sum_{\vec{k}_s}\frac{e^{-i2\pi\vec{n}_s\cdot\vec{k}_s/L}}{\tilde{V}(2\pi\vec{k}_s/L)}. \tag{6}$$

We can rewrite powers of M using an auxiliary field ϕ,

$$M^{L_t} = \int D\phi e^{-S(\phi)}M_{L_t-1}(\phi)\times\cdots\times M_0(\phi), \tag{7}$$

where

$$S(\phi) = -\frac{\alpha_t}{2}\sum_{n_t}\sum_{\vec{n}_s,\vec{n}_s'}\phi(\vec{n}_s,n_t)V^{-1}(\vec{n}_s-\vec{n}_s')\phi(\vec{n}_s',n_t), \tag{8}$$

$$M_{n_t}(\phi) \equiv: \exp\left[-\alpha_t H_{\text{free}} - \alpha_t\sum_{\vec{n}_s}U(\vec{n}_s)\rho(\vec{n}_s) + \alpha_t\sum_{\vec{n}_s}\phi(\vec{n}_s,n_t)\rho(\vec{n}_s)\right]:, \tag{9}$$

$$D\phi = \prod_{\vec{k}_s}[-\tilde{V}(2\pi\vec{k}_s/L)]^{-L_t/2}\prod_{\vec{n}_s,n_t}\frac{d\phi(\vec{n}_s,n_t)}{\sqrt{2\pi/\alpha_t}}. \tag{10}$$

Let $f^{(1)}(\vec{n}_s), f^{(2)}(\vec{n}_s), \ldots$ be a complete set of orthonormal real-valued functions of the spatial lattice sites $\vec{n}_s$. We refer to these functions as orbitals. We take $f^{(1)}(\vec{n}_s)$ to be strictly positive but otherwise regard the form for the orbitals to be arbitrary. If the total number of lattice sites is L^d then we have a total of L^d orbitals. We denote a one-particle state with component i in the kth orbital as $|f_i^{(k)}\rangle$.

Let $\mathcal{B}$ and $\mathcal{C}$ be any subsets of the orbital indices. From these we define $|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle$ as the quantum state where each of j components fill orbitals $\mathcal{B}$ and each of the remaining $2N-j$ components fill the orbitals $\mathcal{C}$. The order of the component labels is irrelevant, and so we assume that the first j components fill orbitals $\mathcal{B}$ and last $2N-j$ components fill orbitals $\mathcal{C}$. The total number of fermions in state $|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle$ is $j|\mathcal{B}| + (2N-j)|\mathcal{C}|$, , where $|\mathcal{B}|$ and $|\mathcal{C}|$ are the number of elements in $\mathcal{B}$ and $\mathcal{C}$, respectively.

We define $E_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ as the energy of the lowest energy eigenstate with nonzero inner product with $|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle$. We let $Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ be the expectation value of M^{L_t} for $|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle$,

$$Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}} = \langle \mathcal{B}^j\mathcal{C}^{2N-j}|M^{L_t}|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle. \tag{11}$$

In the limit of large L_t the contribution from the lowest energy eigenstate dominates and therefore

$$E_{\mathcal{B}^j\mathcal{C}^{2N-j}} = -\lim_{L_t\to\infty} \frac{\ln(Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}})}{\alpha_t L_t}. \tag{12}$$

We can write $Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ using the auxiliary field ϕ,

$$Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}} = \int D\phi e^{-S(\phi)}\langle \mathcal{B}^j\mathcal{C}^{2N-j}|M_{L_t-1}(\phi)\times\cdots \times M_0(\phi)|\mathcal{B}^j\mathcal{C}^{2N-j}\rangle. \tag{13}$$

At this point we define matrix elements for the one-particle states,

$$\mathcal{M}_{k',k}(\phi) = \langle f_i^{(k')}|M_{L_t-1}(\phi)\times\cdots\times M_0(\phi)|f_i^{(k)}\rangle. \tag{14}$$

The component index i in Eq. (14) does not matter due to the SU($2N$) symmetry. Each entry of the matrix $\mathcal{M}_{k',k}(\phi)$ is real. We let $\mathcal{M}_\mathcal{B}(\phi)$ be the $|\mathcal{B}|\times|\mathcal{B}|$ submatrix consisting of the rows and columns in $\mathcal{B}$ and let $\mathcal{M}_\mathcal{C}(\phi)$ be the $|\mathcal{C}|\times|\mathcal{C}|$ submatrix for $\mathcal{C}$. Each normal-ordered transfer matrix operator $M_{n_t}(\phi)$ has only single-particle interactions with the auxiliary field and no direct interactions between particles. Therefore it follows that

$$Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}} = \int D\phi e^{-S(\phi)}[\det\mathcal{M}_\mathcal{B}(\phi)]^j[\det\mathcal{M}_\mathcal{C}(\phi)]^{2N-j}. \tag{15}$$

The form for $Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ in Eq. (15) suggests a simple upper bound based on the Hölder inequality. We recall that the Hölder inequality states that for any integrable functions $f(x)$ and $g(x)$ and positive numbers p and q such that $1/p + 1/q = 1$,

$$\int dx|f(x)g(x)| \le \left[\int dx|f(x)|^p\right]^{1/p}\left[\int dx|g(x)|^q\right]^{1/q}. \tag{16}$$

Let n_1 and n_2 be integers such that $0\le 2n_1 < j < 2n_2 \le 2N$. Let us define the new positive-definite measure

$$\tilde{D}\phi = D\phi e^{-S(\phi)}[\det\mathcal{M}_\mathcal{B}(\phi)]^{2n_1}[\det\mathcal{M}_\mathcal{C}(\phi)]^{2N-2n_2}, \tag{17}$$

so that

$$Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}} = \int \tilde{D}\phi[\det\mathcal{M}_\mathcal{B}(\phi)]^{j-2n_1}[\det\mathcal{M}_\mathcal{C}(\phi)]^{2n_2-j}. \tag{18}$$

We now use the Hölder inequality with $p=(2n_2-2n_1)/(j-2n_1)$, $q=(2n_2-2n_1)/(2n_2-j)$, $dx\to\tilde{D}\phi$, $|f(x)|\to|\det\mathcal{M}_\mathcal{B}(\phi)|^{j-2n_1}$, and $|g(x)|\to[\det\mathcal{M}_\mathcal{C}(\phi)]^{2n_2-j}$. We conclude that $|Z^{L_t}_{\mathcal{B}^j\mathcal{C}^{2N-j}}|$ is bounded above by

$$\left[\int\tilde{D}\phi|\det\mathcal{M}_\mathcal{B}(\phi)|^{2n_2-2n_1}\right]^{(j-2n_1)/(2n_2-2n_1)} \times\left[\int\tilde{D}\phi|\det\mathcal{M}_\mathcal{C}(\phi)|^{2n_2-2n_1}\right]^{(2n_2-j)/(2n_2-2n_1)} = (Z^{L_t}_{\mathcal{B}^{2n_2}\mathcal{C}^{2N-2n_2}})^{(j-2n_1)/(2n_2-2n_1)}(Z^{L_t}_{\mathcal{B}^{2n_1}\mathcal{C}^{2N-2n_1}})^{(2n_2-j)/(2n_2-2n_1)}. \tag{19}$$

Taking the limit $L_t\to\infty$ we deduce that the energies satisfy the inequality

$$E_{\mathcal{B}^j\mathcal{C}^{2N-j}} \ge \frac{j-2n_1}{2n_2-2n_1}E_{\mathcal{B}^{2n_2}\mathcal{C}^{2N-2n_2}} + \frac{2n_2-j}{2n_2-2n_1}E_{\mathcal{B}^{2n_1}\mathcal{C}^{2N-2n_1}}. \tag{20}$$

This is a statement of convexity for $E_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ as a function of j between even end points $j=2n_1$ and $j=2n_2$. We recall that $E_{\mathcal{B}^j\mathcal{C}^{2N-j}}$ is the energy of the state with $|\mathcal{B}|$ particles of each component 1 through j and $|\mathcal{C}|$ particles of each component $j+1$ through $2N$. If we now take $|\mathcal{B}| = K+1$ and $|\mathcal{C}| = K$, then the total particle number is $A = 2NK + j$. The inequality in Eq. (20) is precisely the convexity pattern in Fig. 1 for $E(A)$ as a function of particle number.

We point out that for the special case $K=0$, we can take $\mathcal{B}$ to be the first orbital and $\mathcal{C}$ to be the empty set. In this case $\mathcal{M}_\mathcal{B}(\phi)$ is simply a number. Furthermore, since $f^{(1)}(\vec{n}_s)$ is strictly positive, $\mathcal{M}_\mathcal{B}(\phi)$ is also positive so long as the temporal lattice step a_t is not excessively large. Since $\det\mathcal{M}_\mathcal{B}(\phi) = \mathcal{M}_\mathcal{B}(\phi) > 0$ it is no longer necessary that the power of $\det\mathcal{M}_\mathcal{B}(\phi)$ be even to ensure positivity. Therefore $E(A)$ is actually convex for all A between 0 and $2N$ and not just even A.

These convexity relations could be checked using any number of attractive SU($2N$) models in various dimensions. Here we examine actual nuclear physics data to investigate Wigner's approximate SU(4) symmetry in light nuclei. It is by no means clear that the interactions of nucleons in light nuclei can be approximately described by an attractive SU(4)-symmetric potential. Recent results from nuclear lattice simulations hint that this might be possible [11,12]; however, there are forces even at lowest order in chiral effective field theory which break SU(4) invariance in addition to being repulsive. Nevertheless, all of the SU(4) convexity constraints are in fact satisfied for the most stable light nuclei with up to 16 nucleons, as can be seen in Fig. 2. The line segments drawn show all of the convexity lower bounds.

There have been several recent studies of alpha clustering in nuclear matter [13] as well as multiparticle clustering in other systems [7,8,14]. The results presented here give sufficient conditions for the onset of this multiparticle

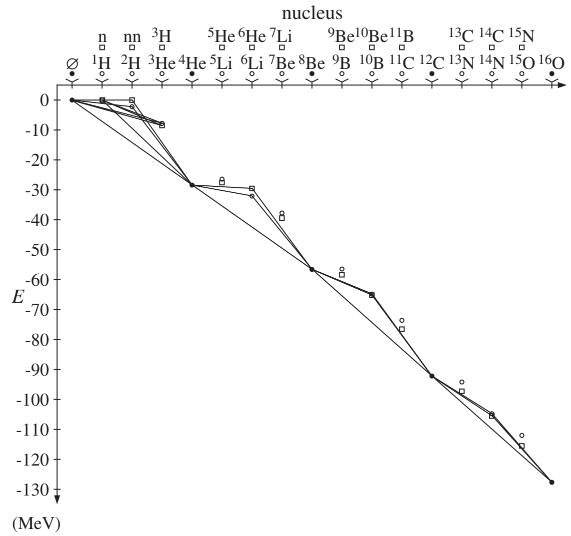

FIG. 2. Plot of the energy versus particle number for the most stable light nuclei with up to 16 nucleons. The line segments show the convexity lower bounds.

clustering phase. One can also make a definite prediction about the j-component quasiparticle energy gaps. Starting from a $2NK$-fermion SU($2N$)-symmetric state, let δ_j be the extra energy required per fermion to add j fermions, all of different components. The ground state energy for $2NK + j$ fermions is a convex function for even j in the interval from $j = 0$ to $j = 2N$. Therefore it follows that $\delta_2 \geq \delta_4 \geq \cdots \geq \delta_{2N}$. Since the ground state energy for $2NK + j$ fermions is also convex for $j = 0, , 1, 2$, we conclude furthermore that $\delta_1 \geq \delta_2 \geq \delta_4 \geq \cdots \geq \delta_{2N}$. We note that for the strongly attractive case these energy gaps are negative, and it is more natural to speak of energy gaps per missing fermion for the corresponding j-component quasiholes, δ^h_j. In this case we find again $\delta^h_1 \geq \delta^h_2 \geq \delta^h_4 \geq \cdots \geq \delta^h_{2N}$.

In summary, we have derived a general result on spectral convexity with respect to particle number for $2N$ degenerate components of fermions. We assume only that the interactions are governed by an SU($2N$)-invariant two-body potential whose Fourier transform is negative definite. The ground state energy E as a function of the number of particles A is convex for even A modulo $2N$. Also $E(A)$ for odd A is bounded below by the average of the two neighboring even values, $E(A-1)$ and $E(A+1)$. When applied to light nuclei for $A \leq 16$ all of the convexity bounds for SU(4) are satisfied. These results give further evidence that an approximate description of light nuclei may be possible using an attractive SU(4)-symmetric potential. This would be a direction worth pursuing since the same theory could then be applied to dilute neutron-rich matter with a finite number of protons. The residual $SU(2) \times SU(2)$ symmetry for proton spins and neutron spins would guarantee that the Monte Carlo simulation could be done without fermion sign oscillations. The physics of this quantum system would be helpful in understanding the superfluid properties of dilute neutron-rich matter in the inner crust of neutron stars.

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