Robust Controlled-NOT Gates from Almost Any Interaction

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There are many cases where the interaction between two qubits is not precisely known, but single-qubit operations are available. In this Letter we show how, regardless of an incomplete knowledge of the strength or form of the interaction between two qubits, it is often possible to construct a controlled-NOT gate which has arbitrarily high fidelity. In particular, we show that oscillations in the strength of the exchange interaction in solid state Si and Ge structures are correctable.

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Any realistic quantum computer has errors. Principally these errors come in two varieties: random decoherence and systematic errors. Systematic errors can arise from imperfections and inhomogeneities in the construction or implementation of demanding experiments. Both systematic errors and errors due to decoherence may be corrected, although it is considerably easier to correct systematic errors.

A pertinent example of systematic error is the strength of the exchange interaction oscillation in solid state silicon based architectures [1–6]. The six conduction-band minima in silicon generate intervalley electronic interference. This causes an unwanted oscillation in magnitude of the exchange splitting between two neighboring donors. Therefore, the strength of the interaction between qubits sensitively depends on the exact positioning of donors. In this Letter we demonstrate that, in principle, systematic errors of this type in the strength or form of interaction between two qubits *are* correctable.

Systematic errors may be corrected using composite pulses, in which a single operation is replaced by several imperfect pulses in such a way that systematic errors in each pulse cancel each other. Freeman [7] and Levitt's review [8] and the references therein provide an excellent introduction. More recently Jones [9] notes that single-qubit composite pulses can modified to apply to the Ising interaction. In particular he presents a two-qubit pulse sequence based on those by Wimperis [10] for the construction of a controlled-NOT (CNOT) gate in NMR.

This Letter applies to any architecture with the ability to apply single-qubit rotations and a coupling between the two qubits. Therefore many leading quantum computing architectures—including solid state architectures—can, in principle, correct for an unknown coupling between qubits. This addresses a common problem across many architectures, where composite pulses have begun to be applied (for example in ion traps [11–13] and Josephson Junctions [14]). As an example, we explicitly consider electron spin in the Kane architecture [15].

Using the method presented here, it is not necessary to know either the *strength* or the *form* of the coupling. We will not assume that the error is in the strength of the interaction alone. In fact, we will demonstrate that it is possible to create a high fidelity CNOT gate from a largely random Hamiltonian.

A key benefit of composite pulses is that the error does not need to be perfectly characterized. Characterizing the strength and form of the interaction to a high degree of accuracy is a challenging task. Even with an accurate characterization of the Hamiltonian, the pulse sequences given in this paper outperform a naive implementation of the CNOT gate. Although we never *learn* the exact Hamiltonian, we arrange that systematic errors cancel themselves.

The composite CNOT gate construction follows the following steps: (i) Isolate a single term: In this step, a single coupling term is isolated from the interaction Hamiltonian. (ii) Create a composite control sign gate: In this step, pulses adapted from NMR correct for systematic errors in the strength of the coupling. (iii) Finally, apply single-qubit unitaries.

A completely general two-qubit Hamiltonian may be expanded in the Pauli basis as

$$H_2 = \sum_{i,j=\{I,X,Y,Z\}} J_{ij}\sigma_i\sigma_j, \qquad (1)$$

where σ_i are the Pauli matrices, and as throughout the Letter, the tensor product is implied. This Hamiltonian includes both coupling terms and single-qubit terms. The coupling energies between the qubits are given by the constants J_{ij} ($i \neq I$, $j \neq I$). We do not assume that we know either the strength of the single-qubit terms, or the coupling terms. There will be a coupling energy which we believe is greatest. Without loss of generality, let us assume that this term is J_{ZZ}. Any single two-qubit term is sufficient.

It is well known that it is possible to isolate a particular term of the interaction using a technique called *term isolation* [16]. In our case, it is possible to isolate the J_{ZZ} term. Consider the pulse sequence

$$Q(t) = Z_\pi^{(1)} Z_\pi^{(2)} V_{t/4} Z_\pi^{(1)} V_{t/4} Z_\pi^{(1)} Z_\pi^{(2)} V_{t/4} Z_\pi^{(1)} V_{t/4}, \qquad (2)$$

$$V_t = X_\pi^{(1)} X_\pi^{(2)} \exp\left(iH_2 \frac{t}{2}\right) X_\pi^{(1)} X_\pi^{(2)} \exp\left(iH_2 \frac{t}{2}\right). \quad (3)$$

Here, as throughout this Letter, a single-qubit rotation of an angle θ around the z axis of the ith qubit is denoted by

$$Z_\theta^{(i)} = \exp\left(i \frac{\theta}{2} \sigma_Z\right), \quad (4)$$

and similarly for rotations around the x and y axes. This pulse sequence isolates a single coupling term

$$Q(t) \approx \exp(iJ_{ZZ} t \sigma_Z \sigma_Z). \quad (5)$$

Equation (5) is only correct to first order, because not all terms in the Hamiltonian, H_2, commute. However, it may be made arbitrarily accurate by applying the pulses, $X_\pi^{(1)} X_\pi^{(2)}$, $Z_\pi^{(1)} Z_\pi^{(2)}$ and $Z_\pi^{(1)}$, k times more frequently: $Q_k(t) = Q(t/k)^k$.

Term isolation is not uniformly valid. If there is no coupling of the specified type (that is $J_{ZZ} = 0$) then the qubits will be decoupled by the pulse sequence, and no term isolated. Also, to perform term isolation it is necessary that the single-qubit rotations are implemented much faster than the typical time scale of the coupling between qubits. This requires either fast single-qubit rotations, or the ability to turn the interaction between qubits on and off.

If interaction Hamiltonian is *known* to have a simpler form, then a single coupling term may be isolated more simply and effectively. For the Heisenberg interaction

$$H_H = J(\sigma_X \sigma_X + \sigma_Y \sigma_Y + \sigma_Z \sigma_Z), \quad (6)$$

all terms commute, and therefore J_{ZZ} can be isolated using just two steps:

$$\exp(iJ_{ZZ} t \sigma_Z \sigma_Z) = Z_\pi^{(1)} \exp(iH_H t) Z_\pi^{(1)} \exp(iH_H t). \quad (7)$$

Equation (7) is exact, and would only need to be carried out once. For many systems, such as the nuclei and electron spins in the Kane architecture, or quantum dots, this much simpler form of term isolation may be used.

This completes the first step: To isolate a coupling single term. For a completely general two-qubit Hamiltonian, it is always possible to isolate a single coupling term. The strength of this term remains unknown, but as this Letter now shows, systematic errors in the strength J_{ZZ} can be corrected.

The exact coupling strength, J_{ZZ} remains unknown. In general we will predict a certain value, J_P. Unless the gate is perfectly characterized, we will make some fractional error, Δ, defined as $J_{ZZ} = (1 + \Delta) J_P$. Therefore, when we attempt to create the gate $\theta_0 = \exp(i \frac{\theta}{2} \sigma_Z \sigma_Z)$, by setting $t = \frac{\theta}{J_P}$ we will systematically overrotate or underrotate, *actually* creating the gate

$$\theta_0^{[1]} = [\theta(1 + \Delta)]_0 \approx Q(t). \quad (8)$$

Jones [9] notes that single-qubit composite pulses can be modified to apply to the Ising interaction. In particular a two-qubit pulse sequence based on BB1 [10] is presented. The symmeterized version of the pulse is

$$\theta_0^{[2]} = (\theta/2)_0^{[1]} \pi_\phi^{[1]} 2\pi_{3\phi}^{[1]} \pi_\phi^{[1]} (\theta/2)_0^{[1]}, \quad (9)$$

where this pulse is made up of imperfect gates,

$$\theta_\phi = Y_\phi^{(2)} \theta_0 Y_{-\phi}^{(2)} \quad (10)$$

and in order to cancel first and second order terms, $\phi = \arccos(-\frac{\theta}{4\pi})$. An alternative pulse which gives the same increase in fidelity when the uncertainty in J_{ZZ} is the only source of error, but which allows us to refocus an additional time, is given by

$$\theta_0^{[2]} = (\theta/2)_0^{[1]} \frac{\pi^{[1]}}{2_\phi} \frac{\pi^{[1]}}{2_{-\phi}} Z_\pi^{(2)} \frac{\pi^{[1]}}{2_\phi} \frac{\pi^{[1]}}{2_{-\phi}} (\theta/2)_0^{[1]} Z_\pi^{(2)}. \quad (11)$$

Pulse schemes on a single qubit may be made arbitrarily accurate [17]. This is also true of two-qubit pulses. One straightforward way to do this is to feed the pulse back into itself. If we implement the pulse sequence,

$$\theta_0^{[2*]} = (X_\pi^{(2)} X_\pi^{(2)} Z_\pi^{(2)} (\theta/16)_0^{[2]} Z_\pi^{(2)} (\theta/16)_0^{[2]})^8, \quad (12)$$

then by feeding this pulse back into the right-hand side of Eq. (11), we obtain a pulse which is correct to higher order. In principle there is no limit to the order which is achievable.

The average fidelity, for the purposes of this Letter, is defined as

$$F = \frac{|\mathrm{Tr}(U_I^\dagger U)|}{\mathrm{Tr}(U^\dagger U)}, \quad (13)$$

where U_I is the actually implemented operation, and U is the intended rotation.

Using each of the three pulse sequences, we attempted to create the entangling component of the CNOT gate, $(\frac{\pi}{2})_0$. Figure 1 shows the fidelity each pulse sequence, plotted against the error, Δ, in the strength of the interaction. The

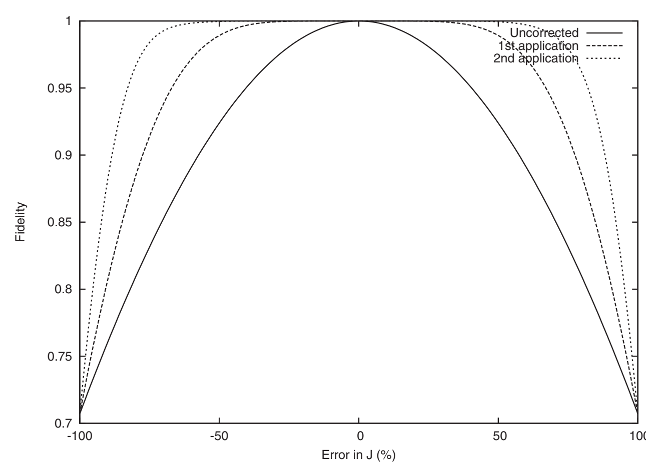

FIG. 1. This plot shows the fidelity of several methods of creating a CNOT gate, with a systematic error in the strength of the coupling.

solid line shows the fidelity without any correction. The first dotted line shows the fidelity of the composite pulse described in Eq. (9) or Eq. (11). The composite pulse provides an improvement over the fidelity of the uncorrected pulse for $J_{ZZ}(1 \pm 100)\%$. The higher order pulse described using Eq. (12) is shown as the second dotted line. It shows an improvement over both the uncorrected pulse and the first composite pulse between $\Delta = -100\%$ and $\Delta = 100\%$.

If we wish to have an error of 1×10^{-4} then without correction we require a $\Delta < 1\%$. For the composite pulse scheme described by Eq. (9) or Eq. (11), we may tolerate an error, $\Delta < 22\%$. In the higher order composite pulse described using Eq. (12), an error $\Delta \approx 41\%$ still achieves a fidelity of 99.99%.

This concludes the second step. A systematic error in the interaction strength, Δ may be corrected using two-qubit extensions of well-known composite pulses. These pulses may be made arbitrarily accurate by concatenation.

For the final step, we simply note that a CNOT gate may be written as

$$\mathrm{CNOT} = H^{(2)} Z^{(1)}_{\pi/2} Z^{(2)}_{\pi/2} \exp\left(i \frac{\pi}{4} \sigma_Z \sigma_Z\right) H^{(2)}. \tag{14}$$

A robust CNOT gate may be constructed applying all three steps. The first step isolates the $\sigma_Z\sigma_Z$ term, regardless of the form of the Hamiltonian. The second step corrects for any error in the strength of this term, and finally the third step applies single-qubit unitaries to complete the *robust* CNOT. Using this robust CNOT gate, we now describe two examples.

One of the current concerns about the viability of the construction of an exchange based solid state quantum computer is oscillations in the strength of the exchange interaction [1,6,18]. For an arbitrarily placed donor, the strength of the exchange interaction is unknown. Even the variation of the donor's position by a single lattice site can change the strength of the exchange interaction dramatically. The placement introduces an unknown systematic error in the strength of the exchange interaction. Fortunately, that is *exactly* the type of error which is corrected in this Letter. It does not matter that we do not know the strength of the interaction, or that the exchange interaction may differ from site to site.

For the Kane quantum computer the single-qubit Hamiltonian is given by

$$H_Q = \mu_B B \sigma_e^z - g_n \mu_n B \sigma_n^z + A(V_A) \sigma_e \cdot \sigma_n, \tag{15}$$

where B is the strength of the constant magnetic field, σ^z is the Pauli Z matrix with subscripts e referring to electrons and n referring to the nucleus and $A(V_A)$ is the strength of the hyperfine interaction. This allows single-qubit operation of the computer using either the nuclear spin as a qubit [15] or electron spin [19]. The exchange coupling between electrons whose strengths, J_i, can vary considerably, leads to the Hamiltonian

$$H = \sum_i J_i \sigma_i \cdot \sigma_{i+1} + H_Q^{(i)}. \tag{16}$$

This Hamiltonian allows for the single-qubit operation of the computer, and also for interaction between the qubits. Using the methods described in this paper, and typical parameters for the Kane architecture, we estimate a corrected CNOT gate between electron spin qubits would require approximately 460 ns to complete. The composite pulse is short compared to the measured decoherence times (as long as 60 ms [20]) of donors in Si.

Assuming perfect single-qubit gates, the fidelities for these two sequences exactly mimic those shown in Fig. 1. In principle, well-known single-qubit composite pulses could be used to create arbitrarily accurate single-qubit gates [17]. The solid, uncorrected curve is extremely sensitive to errors in the strength of the interaction, and therefore also to the exact placement of the donor. The fidelity of the composite pulse follows the first dotted line in Fig. 1. This curve is much less sensitive to errors. As noted above, this pulse improves over the naive case for $\Delta = -1$ to $\Delta = 1$.

We now consider a largely random coupling between qubits. Remarkably, regardless of our incomplete knowledge of the system, we can still create a high fidelity CNOT gate. To demonstrate this, we consider the effect of random systematic error on the fidelity of a CNOT gate. We will assume that the interaction Hamiltonian is given by

$$H_R = J(\sigma_X\sigma_X + \sigma_Y\sigma_Y + \sigma_Z\sigma_Z) + R \sum_{i,j=\{I,X,Y,Z\}} J_{ij}^r \sigma_i \sigma_j. \tag{17}$$

The coefficients J_{ij}^r are chosen uniformly at random between -1 and 1. The factor R gives the strength of the random term in the Hamiltonian. The first three terms in this Hamiltonian give a simple Heisenberg interaction. This nonrandom term in the Hamiltonian represents the interaction or combination of interactions expected to be present in the quantum system. The second, random term in the Hamiltonian contributes to both single-qubit terms, and two-qubit terms. The random term models our incomplete knowledge, not only of the strength of the interaction, but also of the form of the interaction.

We compare the performance of the composite pulse CNOT gate against the well known "square root of swap" construction [21]. Figure 2 shows the average and minimum fidelity of this construction, for different values of R/J. The fidelity for each value of R/J is calculated for 1000 different random Hamiltonians. When the random contribution of the Hamiltonian is large ($R/J = \pm 1$), the uncorrected CNOT gate is useless. It has a average fidelity of approximately 50%. This is worse than if no interaction had been applied at all. Even in the worst case of minimum fidelity, the composite pulse has a fidelity superior to the uncorrected case.

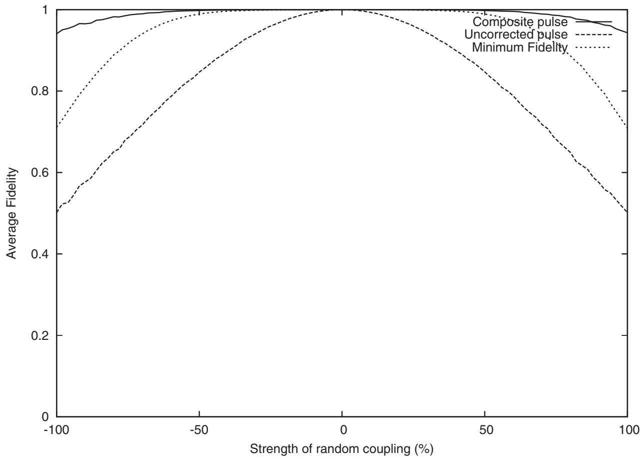

FIG. 2. Graph showing the fidelity of uncorrected and composite pulses to a Hamiltonian with a random component.

If the square root of swap construction is replaced by the composite pulse described in this Letter, on average, a high fidelity CNOT gate may be constructed. The mean fidelity, averaged over 1000 different random Hamiltonians for each value of R/J, is shown as the dotted line in Fig. 2. We also found the minimum fidelity for the composite pulse, and this is also plotted in Fig. 2. Even when the random contribution is as large as the exchange coupling, $R/J = 1$, the average fidelity of the composite gate is approximately 95%.

To obtain this composite pulse scheme, we combined decoupling with a composite pulse scheme. First, the pulse used in Fig. 2 was obtained using Eq. (2). Term isolation was applied using $k = 20$ repetitions for each gate. Second, Eq. (13) was used to correct the strength of the $\sigma_Z\sigma_Z$ term. As Fig. 2 shows, a large increase in fidelity is obtained. Even when the error is large and the coupling between the qubits is essentially random, using the pulse schemes presented in this Letter, it is possible to produce a high fidelity CNOT gate.

In this Letter, we have shown how to create a composite pulse for the CNOT gate. In fact, the method presented here is easily extended to *any* two-qubit gate. A direct method for doing this is to express the gate using the Cartan decomposition for two qubits [22]. By using the composite pulse described here for each of the three nonlocal entangling operations, any two-qubit operation can be made robust to errors in the strength and form of coupling between qubits.

We have presented a composite pulse for creating CNOT gates which corrects for systematic errors in both the form and strength of the interaction between qubits. We applied the composite pulse to a model electron spin architecture. We also considered random systematic errors, showing that systematic errors in the form and strength of the coupling Hamiltonian were corrected. The pulse scheme presented here has broad applicability. Any system which implements single-qubit operations, and has a direct coupling between two qubits directly may perform this implementation of the CNOT gate. We have shown that, regardless of an incomplete knowledge of the strength or form of the interaction between two qubits, in many cases it is possible to construct a CNOT gate which has arbitrarily high fidelity.

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