

Modified Coulomb Law in a Strongly Magnetized Vacuum

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We study the electric potential of a charge placed in a strong magnetic field $B \gg B_0 \simeq 4.4 \times 10^{13}$ G, as modified by the vacuum polarization. In such a field the electron Larmour radius is much less than its Compton length. At the Larmour distances a scaling law occurs, with the potential determined by a magnetic-field-independent function. The scaling regime implies short-range interaction, expressed by the Yukawa law. The electromagnetic interaction regains its long-range character at distances larger than the Compton length, the potential decreasing across \mathbf{B} faster than along. Correction to the nonrelativistic ground-state energy of a hydrogenlike atom is found. In the limit $B = \infty$, the modified potential becomes the Dirac δ function plus a regular background. With this potential the ground-state energy is finite—the best pronounced effect of the vacuum polarization.

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There is now compelling evidence that many compact astronomical objects (soft gamma-ray repeaters, anomalous x-ray pulsars, and some radio pulsars) identified with neutron stars have surface magnetic fields as high as $\sim 10^{14}$ – 10^{15} G [1]. More strong magnetic fields ($B \sim 10^{16}$ – 10^{17} G) are predicted to exist at the surface of cosmological gamma-ray bursters if they are rotation-powered neutron stars similar to radio pulsars [2]. All these fields, however, are much smaller than the maximum value inherent in quantum electrodynamics [3].

Vacuum in an external magnetic field B behaves as an anisotropic dielectric medium with spatial and frequency dispersion, (e.g., [4]). These properties may become important, provided that the field strength achieves the characteristic value $B_0 = m^2/e \simeq 4.4 \times 10^{13}$ G, where m is the electron mass and e is its charge. [Henceforth, we set $\hbar = c = 1$ and refer to the Heaviside-Lorentz system of units.] Although much work has been devoted to the study of electromagnetic wave propagation in the magnetized vacuum, problems of electro- and magnetostatics in this medium did not attract sufficient attention, save Refs. [5,6], where corrections to the Coulomb law were found when these are small: for $B/B_0 \ll 1$ in [5], or at large distances from the source for $1 \ll B/B_0 \ll 3\pi\alpha^{-1}$ in [6] ($\alpha = e^2/4\pi = 1/137$).

In this Letter, we find that for sufficiently large $b \equiv B/B_0 \gg 1$ the electric field produced by a pointlike charge at rest may be significantly modified by the vacuum polarization, the modification being determined by the characteristic factor αb . The modified Coulomb potential in the close vicinity of its charge, characterized by the Larmour length $L_B = (eB)^{-1/2} = (1/m\sqrt{b})$, goes steeper than the standard one, following a Yukawa law, whereas it obeys a long-range “anisotropic Coulomb law” far from the source, at distances characterized by the electron Compton length m^{-1} , $m^{-1} \gg L_B$. (Details of the corresponding derivations can be found in Ref. [7].) The short-range part of the modified potential tends to the Dirac δ

function in the limit $b \rightarrow \infty$. The modification of the Coulomb law should affect, first of all, the field of an atomic nucleus, placed in a magnetic field. We determine the corresponding correction to the lowest energy level of a hydrogenlike atom. Unlike the famous result of Ref. [8], referred to in many speculations on behavior of matter on the surface of strongly magnetized neutron stars (e.g., the review [9] and references therein), we find that the (vacuum-polarization-corrected) ground-state energy remains finite in the limit of infinite magnetic field.

Let the constant and homogeneous magnetic field \mathbf{B} be directed along axis 3 in the frame where the pointlike charge q is at rest in the origin $\mathbf{x} = \{x_1, x_2, x_3\} = 0$, and no external electric field exists. By using the tensor decomposition of the photon propagator over eigenmodes in a magnetic field [10,11] it is straightforward to show [7] that electrostatic potential A_0 produced by this charge has the form

$$A_0(\mathbf{x}) = \frac{q}{(2\pi)^3} \int \frac{\exp(-i\mathbf{k} \cdot \mathbf{x}) d^3k}{\mathbf{k}^2 - \kappa_2(k_3^2, k_\perp^2)}, \quad (1)$$

while its vector potential is zero, $A_{1,2,3}(\mathbf{x}) = 0$. The static charge gives rise to electric field only, as it might be expected. Here κ_2 is one (out of three) eigenvalue(s) of the polarization operator Π_μ^ν : $\Pi_\mu^\nu b_\nu^{(a)} = \kappa_a b_\mu^{(a)}$, $a = 1, 2, 3$, $\mu, \nu = 0, 1, 2, 3$. The eigenvectors $b_\nu^{(a)}$ are 4-potentials of the eigenmodes. The eigenvalues depend on two combinations of the photon momentum components $k_3^2 - k_0^2$ and $k_\perp^2 = k_1^2 + k_2^2$ taken at zero frequency $k_0 = 0$. Equation (1) is approximation independent and axial symmetric: $A_0(\mathbf{x}) = A_0(|x_3|, x_\perp)$, $x_\perp = \sqrt{x_1^2 + x_2^2}$.

Equation (1) indicates that only mode-2 photons mediate electrostatic interaction. This fact may be better understood, if we examine electric and magnetic fields intrinsic to the virtual (off-shell) photons of this mode, obtained from its 4-vector potential $b_\nu^{(2)} = (k_3 \ 0 \ 0 \ k_0)_\nu$. It

may be seen from explicit representation for these fields [11] that in the limit $k_0 = 0$ the magnetic field of mode-2 photon disappears, while its electric field is collinear with \mathbf{k} ; i.e., it becomes a purely longitudinal virtual photon. Virtual photons of other modes, in the static case, are carriers of stationary magnetic fields. For instance, mode-1 photons are responsible for the field produced by a constant current, flowing parallel to the external magnetic field.

In the asymptotic regime of high magnetic field, $eB \gg k_3^2$ and $B \gg m^2/e \equiv B_0$, with the accuracy to terms that only grow with B as its logarithm or slower, the eigenvalue

κ_2 , calculated within the one-loop approximation in [10,11], acquires the form [6,12]

$$\kappa_2(k_3^2, k_\perp^2) = -\frac{2\alpha b m^2}{\pi} \exp\left(-\frac{k_\perp^2}{2m^2 b}\right) T\left(\frac{k_3^2}{4m^2}\right), \quad (2)$$

$$T(y) = y \int_0^1 \frac{(1-\eta^2)d\eta}{1+y(1-\eta^2)}. \quad (3)$$

Note the properties $T(y \rightarrow 0) \simeq 2y/3$, $T(\infty) = 1$.

The deviation of the potential (1) from the standard Coulomb potential $A_0^C(\mathbf{x}) = q/(4\pi\sqrt{x_\perp^2 + x_3^2})$ is

$$\Delta A_0(\mathbf{x}) \equiv A_0^C(\mathbf{x}) - A_0(\mathbf{x}) = \frac{q}{8\pi^2} \int_0^\infty J_0(k_\perp x_\perp) dk_\perp^2 \int_{-\infty}^\infty \left(\frac{\exp(-ik_3 x_3)}{k_\perp^2 + k_3^2} - \frac{\exp(-ik_3 x_3)}{k_\perp^2 + k_3^2 - \kappa_2(k_3^2, k_\perp^2)} \right) dk_3. \quad (4)$$

Here J_0 is the Bessel function of order zero. This integral defines $\Delta A_0(\mathbf{x})$ as a finite function of the coordinates in the origin, unlike A_0 and A_0^C . As the integration variable k_3 in (4) approaches the large values $\pm\sqrt{eB}$, one has $\kappa_2 \ll k_3^2$. Hence, no essential contribution comes from the integration over the region $|k_3| \gtrsim \sqrt{eB}$, wherein Eq. (2) is not valid. The expression to be substituted for it there is even smaller (since it does not contain the large factor b) [13]. The leading terms of the expansion of (1) near the origin

$x_3 = x_\perp = 0$ are

$$A_0(\mathbf{x}) \simeq \frac{q}{4\pi} \left(\frac{1}{|\mathbf{x}|} - 2mC \right), \quad C = \frac{2\pi}{qm} \Delta A_0(0) > 0, \quad (5)$$

where C is a constant depending on the magnetic field.

More exactly, the singular behavior near the origin will be presented if we note that it is provided by the integration over large k_3 (and k_\perp) in (1), where we may set $T(k_3^2/4m^2) = T(\infty) = 1$ and perform k_3 -integration in Eq. (1) by calculating residues. Then

$$A_0(\mathbf{x}) \simeq \frac{\tilde{A}_0(\tilde{\mathbf{x}})}{L_B} = \frac{q}{4\pi L_B} \int_0^\infty J_0(\tilde{k}_\perp \tilde{x}_\perp) \tilde{k}_\perp \frac{\exp[-|\tilde{x}_3| \sqrt{\tilde{k}_\perp^2 + (2\alpha/\pi) \exp(-\tilde{k}_\perp^2/2)}]}{\sqrt{\tilde{k}_\perp^2 + (2\alpha/\pi) \exp(-\tilde{k}_\perp^2/2)}} d\tilde{k}_\perp. \quad (6)$$

Here $\tilde{A}_0(\tilde{\mathbf{x}})$ is a dimensionless, external-field-independent function of the arguments $\tilde{x}_3 = x_3/L_B$, $\tilde{x}_\perp = x_\perp/L_B$, and the integration variables $\tilde{k}_3 = k_3 L_B$, $\tilde{k}_\perp = k_\perp L_B$ are used. We refer to Eq. (6) as establishing a scaling regime that describes the potential measured in inverse Larmor units as a universal function of coordinates measured in Larmor units. It holds for $|\mathbf{x}| \ll (2m)^{-1}$.

The simple representation (6) can be further simplified if x_3 or x_\perp are large in the Larmor scale: $|\tilde{x}_3| \gg 1$, or $|\tilde{x}_\perp| \gg 1$ (but remain small in the Compton scale). In this case the integration in (6) is restricted to the domain $\tilde{k}_\perp^2 \ll 1$, where the exponential $\exp(-\tilde{k}_\perp^2/2)$ should be taken as unity. Then (6) is reduced to the isotropic Yukawa law

$$A_0(\mathbf{x}) \simeq \frac{q}{4\pi L_B} \frac{\exp(-(2\alpha/\pi)^{1/2} \sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2})}{\sqrt{\tilde{x}_\perp^2 + \tilde{x}_3^2}}. \quad (7)$$

This can be established by tracing (6) back to (1) with

$$-\kappa_2(\infty, 0) = \frac{2\alpha}{\pi L_B^2} = \frac{2\alpha b}{\pi} m^2 = M^2 \quad (8)$$

substituted for $-\kappa_2(k_3^2, k_\perp^2)$ in the denominator. Here M is the ‘‘effective photon mass’’ noted in Ref. [15]. The Yukawa law (7) establishes the short-range character of the static electromagnetic forces in the Larmor scale. We

stress, however, that the genuine photon mass understood as its rest energy is always strictly equal to zero as a consequence of the gauge invariance reflected in the approximation-independent relation $\kappa_a(0, 0) = 0$ respected by (2). Hence, the potential, produced by a static charge, should be long range for sufficiently large distances. This is the case, indeed. One can see by inspecting the curves of Fig. 1, computed using Eq. (1) for $x_\perp = 0$, that the scaling regime (6), or (7), does fail for sufficiently large values of x_3 : $\sim 0.1/(2m) = 50L_B$ for $b = 10^6$, $\sim 0.2/(2m) = 33L_B$ for $b = 10^5$, $\sim 0.3/(2m) = 15L_B$ for $b = 10^4$ —the larger, the smaller the field [for these distances Eqs. (6) and (7) are already the same]. Starting with these values, the potential curves approach their envelope, that can be fitted as $A_0(x_3, 0) \simeq (q/4\pi) 1.41/(x_3 + 1.04/2m)$, unlike the scaling curves (6) and (7) that tend fast to zero. It is in this place that the abruptly falling—short-range—potential turns into a slowly decreasing—along the envelope curve—long-range potential. An analogous change from the short- to long-range behavior is observed in Fig. 2, where the electron energy in the field (1) is plotted against the transverse distance from the charge at $x_3 = 0$. At small distances, all the dashed curves in Figs. 1 and 2 approach the thick solid Coulomb curve in accord with (5).

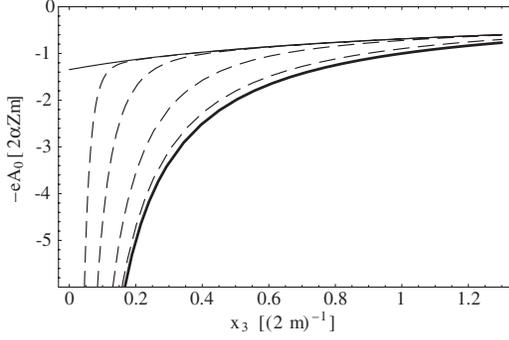


FIG. 1. Electron potential energy $-eA_0$ in the field of a point charge $q = Ze$ plotted in units $2\alpha Zm = Z \times 7.46$ keV as a function of longitudinal coordinate x_3 in Compton half-lengths $(2m)^{-1}$ at $x_\perp = 0$ for four values of magnetic field (from left to right): $b = 10^6, 10^5, 10^4, 10^3$, $b = B/B_0$, $B_0 = 4.4 \times 10^{13}$ G (dashed lines). Solid line is the fit $-1.4/(2mx_3 + 1.04)$. Bold line is the Coulomb law $-1/2mx_3$.

The scaling regime depends on the fact that the eigenvalue (2) grows linearly with the magnetic field. If this linearity, supposedly, retains in higher-loop approximations, one may conjecture that the calculations of the latter may be reduced to finding α^n corrections to the mass (8). Potential (6) implies the suppression of electrostatic force by the linearly growing term in the denominator of (1) at large distances in the Larmor scale from the charge, but not close to the charge, where it has the same singularity as the Coulomb law.

For larger distances x_3, x_\perp , not small in the Compton scale, one may expect that only integration over small k_3, k_\perp is important in (1) (for a more thorough analysis see [7]). In this limit (2) behaves as $\kappa_2(k_3^2, 0) \simeq -\frac{\alpha b}{3\pi} k_3^2$. With this substitution the integral in (1) is

$$A_0(x_3, x_\perp) \simeq \frac{1}{4\pi} \frac{q}{\sqrt{(x'_\perp)^2 + x_3^2}}, \quad (9)$$

where $x'_\perp = \beta x_\perp$, $\beta \equiv (1 + \alpha b/3\pi)^{1/2}$, $x'_\perp > x_\perp$. For small $\alpha b/3\pi$, Eq. (9) coincides with the result of [6].

Equation (9) is an ‘‘anisotropic Coulomb law’’, according to which the attraction force decreases with distance

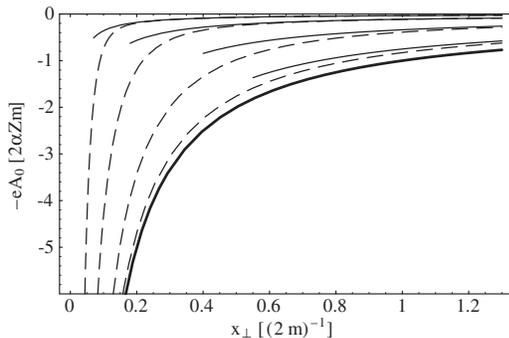


FIG. 2. The same as in Fig. 1 in function of transverse coordinate x_\perp at $x_3 = 0$. The asymptotes at large x_\perp values, $-1/2mx'_\perp$, are presented by solid lines.

from the source along the transverse direction faster than along the magnetic field, but remains long range. In accord with (9), the curves $A_0(x_3, 0)$ in Fig. 1 all approach at large distances $|x_3|$ the Coulomb law $q/(4\pi|x_3|)$, whereas each curve $A_0(0, x_\perp)$ in Fig. 2 reaches at large x_\perp the asymptote $A_0(0, x_\perp) = q/(4\pi x'_\perp) = A_0^C(0, x_\perp)/\beta$, different for each field; i.e., the potential is anisotropic. Again, the same as for short distances considered above, we face—now anisotropic—suppression of the Coulomb force due to the linearly growing term in (2). The equipotential surface is an ellipsoid stretched along the magnetic field. The electric field of the charge $\mathbf{E} = -\nabla A_0(x_3, x_\perp)$ is a vector with the components $(q/4\pi)(x_3^2 + \beta^2 x_\perp^2)^{-3/2}(x_3, \beta^2 x_\perp)$. It is not directed towards the charge, but makes an angle ϕ with the radius-vector \mathbf{r} , $\cos\phi = (x_3^2 + \beta^2 x_\perp^2)(x_3^2 + \beta^4 x_\perp^2)^{-1/2} \times (x_3^2 + x_\perp^2)^{-1/2}$. In the limit of infinite magnetic field, $\beta = \infty$, the electric field of the point charge is directed normally to the axis x_3 .

The regime (9) corresponds to the approximation, where only quadratic terms in powers of the photon momentum are kept in κ_2 . Within this scope the dielectric permeability of the vacuum is independent of the frequency, and the refractive index depends only upon the angle in the space [16].

Consider an impact of the Coulomb potential modification on the nonrelativistic hydrogenlike atom ground-state energy. The wave function $\Psi(x_3)$ is subject to the one-dimensional Schrödinger equation [8] with respect to x_3 , valid in the region $|x_3| > L_B$. The transverse coordinate x_\perp in the argument of the potential is replaced by the momentum of the transverse center-of-mass motion [17] and should be set equal to zero for the ground state.

For the fields in the range $1 \ll b \ll 2\pi/\alpha \sim 10^3$ the correction (4) to the Coulomb potential may be treated as perturbation. Keeping only the first-order term in κ_2 in (4) we may derive the magnetic analog of the Uehling potential [14]. At $x_\perp = 0$ it is

$$\Delta A_0(x_3, 0) \simeq \frac{q\alpha b m}{8\pi^2} \int_0^{\pi/2} e^{-(2m|x_3|/\cos\phi)} \cos^2\phi d\phi. \quad (10)$$

This correction is of the order of αb , i.e., is much larger than α . Equation (10) implies $C \simeq \alpha b/16$. Calculating its matrix element with Loudon’s [8] wave functions we obtain the positive correction ($q = Ze$)

$$\frac{2Z^2\alpha^3 b m}{3\pi} \ln \frac{b}{4\alpha^2} = Z^2 b \left(\frac{\ln b}{8.454} + 1 \right) \times 0.356 \text{ eV} \quad (11)$$

to the negative Loudon-Elliott ground-state energy

$$E_0 = -2Z^2\alpha^2 m \ln^2 \frac{\sqrt{b}}{2\alpha}. \quad (12)$$

Equation (11) may be used for $Z \leq 6$. For instance, for $Z = 1$ and $b = 200$ it makes about 100 eV, while $E_0 \approx -2.5$ keV. Although the proton has a finite size $R \sim 10^{-13}$ cm, the Coulomb $1/|x_3|$ part of the potential (5) remains the same within the definition range of the

Schrödinger equation, since $R < L_B$ within the range of b considered in this paragraph (moreover, $R \ll L_B$). On the contrary, the vacuum-polarization part does depend on R , like it does in the no-magnetic-field case [18]. However, the finite-size correction to (11) is $\sim (R/L_B)^2$ to be neglected within the present scope of accuracy.

The singularity $1/|x_3|$ of the Coulomb potential in the origin is known to lead to the energy spectrum unbounded from below [8]: as this singularity is cut off at the Larmour length, $L_B = (m\sqrt{b})^{-1}$, the ground-state energy (12) tends to $-\infty$ with the growth of the magnetic field, when $L_B \rightarrow 0$. This feature is cured by the vacuum polarization. As $b \rightarrow \infty$ the region in Fig. 1 between the potential curve and the abscissa below the point $-eV \approx -1.4 \times 2\alpha Zm$, where it is crossed by the envelope, becomes infinitely deep and thin. The area $S = (4\pi/q) \int_{L_B}^{\bar{x}_3} A_0(x_3, 0) dx_3$ calculated with expression (5), where \bar{x}_3 is found from the equation $A_0(\bar{x}_3, 0) = V$, has a finite limit if and only if C grows proportionally to $1/L_B$. This is the case: when calculated following Eq. (6), $C \approx 0.9594\sqrt{\alpha b/2\pi}$. Thus, the dominating behavior of the modified Coulomb potential in the origin becomes the δ function. Combining it with the fit for the envelope we come to the limiting form of the electron potential energy at $b = \infty$ [here $|x_3|$ does not exceed a few $(2m)^{-1}$]

$$-eA_0(x_3, 0) = -2\alpha Z \left(S\delta(x_3) + \frac{1.4m}{2m|x_3| + 1.04} \right), \quad (13)$$

where $S = \ln(\sqrt{\pi/2\alpha}/0.96) - 1 + 0.96\sqrt{2\alpha/\pi} = 1.79$. A more rigorous calculation done with the use of Eq. (6) for the short-range part of the potential instead of (5) results in $S \approx -\text{Ei}[-(2\alpha/\pi)^{1/2}] \approx 2.18$; Ei is the exponential integral. With the $1/|x|$ singularity replaced by the δ function, the ground energy level is certainly finite. Applying the formula for the ground energy $-2m(\int_{L_B}^{a_B} A_0(x_3, 0) dx_3)^2$ valid in a shallow well potential [19], with Eq. (13) integrated up to the point $x_3 = 2.6/2m$, where the fitted curve crosses the Coulomb law, and the latter taken as the integrand for larger x_3 up to the Bohr radius $a_B = (m\alpha)^{-1}$, we estimate the *finite* limiting value for the ground energy as

$$E_{\text{lim}} = -2mZ^2\alpha^2 73.8 = -Z^2 \times 4 \text{ keV}. \quad (14)$$

The Loudon-Elliott energy (12) would overrun the limiting energy (14) already for the magnetic field as large as $b = 6600$, when yet $R \ll L_B$. The ground level reaches 92% of its limiting value for $b = 5 \times 10^4$. After the magnetic field reaches the value $b = 1.5 \times 10^5$, when R and L_B equalize, the Coulomb potential is cut off at the proton size, $x_3 = R$. Setting $L_B = R$ in (12) we would get the minimum value for the Loudon-Elliott energy ($Z = 1$) to be -5.6 keV , which is essentially lower than (14).

The significant modification of the Coulomb potential of an electric charge by the vacuum polarization in external constant magnetic field $B \gg B_0$, shown to eliminate the

unboundedness from below of the nonrelativistic hydrogen spectrum, is apt of having more implementation as far as other situations where electrostatic fields are important are concerned, e.g., properties of matter on the surfaces of extremely magnetized neutron stars.

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