Driven Resonance in Partially Relaxed Plasmas

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A Taylor-relaxed plasma ($\mathbf{j} = k\mathbf{B}$ with k a constant) under external magnetic helicity injection encounters resonances in spatial frequencies of its force-free eigenmodes. Such driven resonance underlies the physics of magnetic self-organization and determines the flux amplification in laboratory helicity injection applications. Here we show that for partially relaxed plasmas where the deviation from the fully relaxed Taylor state, for example, a flux-dependent k, is a function of the normalized flux χ/χ_a with χ_a the poloidal flux at the magnetic axis, a modified driven resonance persists even if $k(\chi)$ has an order-unity variation across the flux surfaces.

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Magnetic relaxation was first suggested by Taylor [1] as the cause of the spontaneous formation of a quiescent reversed field pinch configuration in the laboratory toroidal pinch experiments. An important and robust laboratory application of magnetic relaxation [2] has been to form plasma confining magnetic fields such as those of spheromak [3] and spherical tokamak [4] by external magnetic helicity injection. This is carried out by electrically biasing open magnetic field lines that intercept the electrodes and hence inducing a primarily parallel current flowing along the open field lines with a nominal $(j_\parallel/B)_{\rm inj}$ [5]. The open field line kinks driven by the open field line current [6] set off a helical instability cascade in space [7] that tends to flatten $j_\parallel/B$ throughout the plasma [8]. Since the helical fluctuations that facilitate global relaxation usually have a much smaller energy content in comparison with the axisymmetric component of the magnetic field [7,9], it is often useful to examine the toroidally averaged magnetic field alone and speak of a $(j_\parallel/B)_{\rm int}$ inside the separatrix of the mean field, i.e., the field lines of the mean field that do not connect to the electrodes. In the limit of Taylor relaxation [1,2], $(j_\parallel/B)_{\rm inj} = (j_\parallel/B)_{\rm int} = k$ is a global constant.

The reason that the injected magnetic energy and helicity are expected to self-organize onto the device-scale spheromak and spherical tokamak magnetic fields is the result of a resonance phenomenon when k approaches the eigenvalues of the intrinsic Chandrasekhar-Kendall and Yoshida-Giga modes of the discharge chamber [10–12]. For a given magnetic helicity, the relaxed state of $k^2 < k_1^2$, compared with that of $k^2 > k_1^2$, always has the lower magnetic energy [10]. For laboratory spheromak applications, k_1 is the eigenvalue of the first axisymmetric Chandrasekhar-Kendall mode that carries a net toroidal flux [13], so the minimum magnetic energy state bounded by $k^2 < k_1^2$ has a magnetic field on the spatial scale of the entire discharge chamber ($\gtrsim 1/k_1$). From an operational perspective, as the injector and plasma currents are ramped up, the initial increase in $k = (j_\parallel/B)_{\rm inj}$ because of higher $j_\parallel$, saturates to the limit of k_1 because of a diverging B that is proportional to the plasma current. This is the plasma response to a resonant barrier at $k = k_1$ that a Taylor-relaxed spheromak plasma cannot be driven past, and represents an extreme form of the self-organization phenomenon that transfers and accumulates the input energy into large scale coherent structures. An obvious practical importance of the driven resonance in Taylor relaxation is the predicted access to high magnetic field strength spheromak that is achieved through high flux amplification and hence high current multiplication. The significance of a driven resonance in astrophysical magnetic self-organization in the radio lobes has been elucidated earlier [11].

The fully relaxed constant k Taylor state is known to be an ideal limit since realistic plasma in the laboratory appears to always show an incomplete relaxation where, for example, k varies across the plasma. The simplest model for describing such partial relaxation is a force-free plasma where $\nabla \times \mathbf{B} = k(\chi)\mathbf{B}$ and $k(\chi)$ is a flux function, for example,

$$k(\chi) = k_c(1 - \epsilon\chi/\chi_{\rm inj}), \tag{1}$$

with ϵ parametrizing its deviation from the fully relaxed Taylor state. In the more general case, the linear dependence can be replaced by an arbitrary polynomial of $\chi/\chi_{\rm inj}$. Both Kitson and Browning [14] and Tang and Boozer [11] considered these models. It is shown [11] that the nonlinearity introduced by $k(\chi) = k_c(1 - \epsilon\chi/\chi_{\rm inj})$ into the force-free Grad-Shafranov equation regularize the linear resonance. Moreover, the deviation from the Taylor state, represented by a nonvanishing ϵ, imposes an upper bound on the flux amplification factor $\mathcal{A}_F \equiv \chi_{\rm int}/\chi_{\rm inj}$, which occurs for $k_c > k_1 + O(\epsilon)$,

$$\mathcal{A}_F^{\rm sp} = \frac{2}{3}\frac{k_c^2 - k_1^2}{k_c^2}\frac{\langle\chi_1^2\rangle}{\langle\chi_1^3\rangle}\frac{\chi_{\rm sp}}{\epsilon} \tag{2}$$

with $\chi_{\rm sp}$ the poloidal flux in the χ_1 mode of eigenvalue k_1 and $\langle\cdots\rangle \equiv (1/2\pi)\int(\cdots/R^2)d^3\mathbf{x}$. Should the plasma develop a normalized parallel current density that varies significantly across the flux surfaces, e.g., ϵ approaches order unity in Eq. (1), the flux amplification potential would be severely limited.

In this Letter, we will describe a situation where a driven resonance persists even if $k(\chi)$ has an order-unity variation across the flux surfaces. This opens a possibility for high field spheromaks via high flux amplification and hence high current multiplication in the experimentally more relevant, partially relaxed plasmas. The requirement is for $k(\chi)$ to be a function of χ/χ_a where χ_a is the poloidal flux at the magnetic axis. The simplest model, as a counterpart to Eq. (1), is [15]

$$k(\chi) = k_c(1 - \epsilon\chi/\chi_a). \tag{3}$$

It can be argued that the flux dependence of $k(\chi)$ in a high temperature relaxed plasma should be normalized by the instantaneous flux (e.g., χ_a) rather than a preset one (e.g., $\chi_{\rm inj}$). In essence, for a high temperature plasma where plasma fluctuations tend to flatten $j_\parallel/B$ according to the mean field Ohm's law [8], an initial plasma of Eq. (1) at $k_c < k_1$ implies a relaxed plasma of Eq. (3) when k_c reaches beyond k_1 by order ϵ^0. This comes about because, as k_c is beyond k_1 by an amount greater than order ϵ where the flux amplification is given in Eq. (2), Eq. (1) predicts a hollow current density profile in the small ϵ limit that is itself independent of ϵ,

$$k(\chi) = k_c\left(1 - \frac{2}{3}\frac{k_c^2 - k_1^2}{k_c^2}\frac{\langle\chi_1^2\rangle}{\langle\chi_1^3\rangle}\frac{\chi_{\rm sp}}{\chi_{\rm inj}}\chi_1\right).$$

Such ϵ-independent hollow current profile is subject to further relaxation that is represented by the diffusive term of $j_\parallel/B$ in the mean field Ohm's law [8]. The result would be a form of Eq. (3) which models an order-ϵ deviation from the Taylor state throughout the plasma.

Our analysis is based on the nonlinear force-free Grad-Shafranov equation,

$$\Delta^*\chi + G dG/d\chi = 0, \tag{4}$$

for an axisymmetric force-free magnetic field $\mathbf{B} = G\nabla\varphi + \nabla\chi \times \nabla\varphi$ with $k(\chi) = -dG/d\chi$. The new physics will be illustrated with a $k(\chi)$ given in Eq. (3), where $0 < \epsilon < 1$ denotes a partially relaxed plasma that has a higher edge k than its core values. This Letter focuses on the simply connected geometry (e.g., spheromak and radio lobes), with the understanding that the results can be generalized to a torus (field reversed pinch and spherical tokamak under coaxial helicity injection) where the constrained resonance [12] appears in place of the unconstrained Jensen-Chu resonance [10] due to toroidal flux conservation. Since $k(\chi) = -dG/d\chi$, one finds that in a simply connected geometry,

$$G = -k_c\left(\chi - \frac{1}{2}\epsilon\frac{\chi^2}{\chi_a}\right).$$

The force-free Grad-Shafranov equation is

$$\Delta^*\chi + k_c^2\left(1 - \epsilon\frac{\chi}{\chi_a}\right)\left(\chi - \frac{1}{2}\epsilon\frac{\chi^2}{\chi_a}\right) = 0. \tag{5}$$

All the numerical results presented in this Letter are obtained by directly solving Eq. (5) on a grid using a multilevel Newton's method in a cylindrical chamber of unity elongation.

With a fixed vacuum bias flux χ_v as in Fig. 2 of Ref. [11], we numerically solve Eq. (5) and compute the magnetic energy as a function of k_c for different ϵ's. Instead of a regularized resonance for finite ϵ as in Fig. 3 of Ref. [11], we find that a modified resonance persists at a shifted k_c; Fig. 1. It must be noted that this new form of resonance is distinctly different from another case considered by Kitson and Browning [14] in which k is a fixed function of position rather than flux. The latter case subjecting to a linear resonance has a straightforward interpretation since Eq. (4) is formally still a linear equation if k is a known function of position. In our case where k is a flux function, Eq. (3), the implicit dependence of k on position varies as the plasma approaches the resonance, along with the q profile; Fig. 2. It is indeed a nonlinear system that finds its way to lock onto a resonance as a single parameter k_c is being tuned.

This new form of modified resonance can be theoretically understood. The analytical calculation is carried out on the leading order expansion of Eq. (5) in ϵ, which has the form

$$\Delta^*\chi + k_c^2\chi\left(1 - \frac{3}{2}\epsilon\frac{\chi}{\chi_a}\right) = 0.$$

Next, we need to express χ_a as an explicit function of χ to facilitate further analytical analysis. A convenient choice is to write

$$\chi_a = g k_1\langle\chi\rangle,$$

with g an order-unity factor that might change slowly with k_c, but which we will ignore. Let

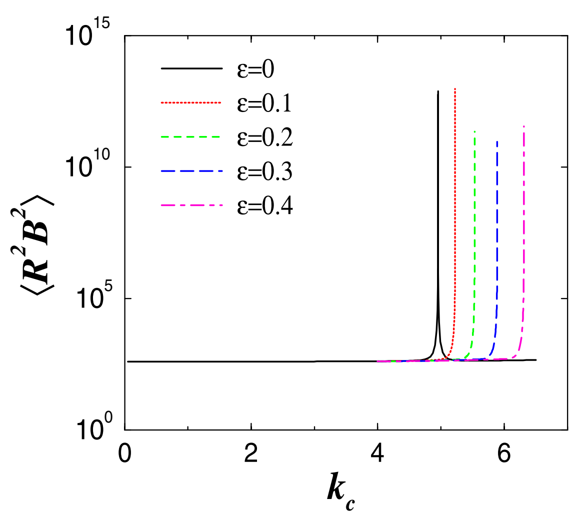

FIG. 1 (color online). Volume integrated magnetic energy is plotted as a function of k_c in the first order model $k = k_c(1 - \epsilon\chi/\chi_a)$ as the partially relaxed plasma approaches the modified resonance for $\epsilon = 0, 0.1, 0.2, 0.3, 0.4$.

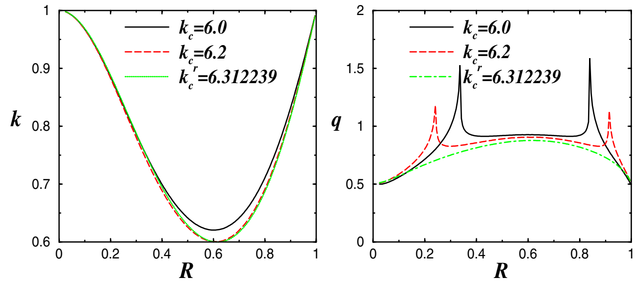

FIG. 2 (color online). The $k(\chi)$ (left) and $q(\chi)$ (right) profiles at two k_c's below the modified resonance are plotted as functions of the major radius R in the midplane, along with those of the modified resonant mode. These are at $\epsilon = 0.4$ in the first order model.

$$\epsilon' \equiv \frac{3\epsilon}{2gk_1},$$

one has a form that is amiable to analysis (dropping the prime),

$$\Delta^*\chi + k_c^2\chi\left(1 - \epsilon\frac{\chi}{\langle\chi\rangle}\right) = 0. \tag{6}$$

Substituting $\chi = \chi_v + \sum_i \alpha_i\chi_i$ into Eq. (6) where $\Delta^*\chi_v = 0$ with boundary condition $\chi_v|_{\partial\Omega} = \chi|_{\partial\Omega}$ and $\Delta^*\chi_i + k_i^2\chi_i = 0$ with $\chi_i|_{\partial\Omega} = 0$, one has

$$-\sum_i \alpha_i k_i^2\chi_i + k_c^2\left(\chi_v + \sum_i \alpha_i\chi_i\right) \times \left[1 - \frac{\epsilon}{\langle\chi\rangle}\left(\chi_v + \sum_j \alpha_j\chi_j\right)\right] = 0.$$

Multiplying this by χ_n, then applying $\langle\cdots\rangle$, one has

$$(k_c^2 - k_n^2)\langle\chi_n^2\rangle\alpha_n\langle\chi\rangle - 2\epsilon k_c^2\sum_i\langle\chi_i\chi_n\chi_v\rangle\alpha_i - \epsilon k_c^2\sum_{ij}\langle\chi_i\chi_j\chi_n\rangle\alpha_i\alpha_j + k_c^2\langle\chi_v\chi_n\rangle\langle\chi\rangle - \epsilon k_c^2\langle\chi_v^2\chi_n\rangle = 0.$$

Much can be understood by truncating the expansion series at $i = 1$, so $\chi = \chi_v + \alpha_1\chi_1$ and $\langle\chi\rangle = \langle\chi_v\rangle + \alpha_1\langle\chi_1\rangle$. We then have

$$(k_c^2 - k_1^2)\langle\chi_1^2\rangle\alpha_1(\langle\chi_v\rangle + \alpha_1\langle\chi_1\rangle) - 2\epsilon k_c^2\langle\chi_1^2\chi_v\rangle\alpha_1 - \epsilon k_c^2\langle\chi_1^3\rangle\alpha_1^2 + k_c^2\langle\chi_v\chi_1\rangle(\langle\chi_v\rangle + \alpha_1\langle\chi_1\rangle) - \epsilon k_c^2\langle\chi_v^2\chi_1\rangle = 0.$$

Collecting the terms in the power of α_1, one finds a quadratic equation

$$a\alpha_1^2 + b\alpha_1 + c = 0,$$

with

$$a = (k_c^2 - k_1^2)\langle\chi_1^2\rangle\langle\chi_1\rangle - \epsilon k_c^2\langle\chi_1^3\rangle;$$
$$b = (k_c^2 - k_1^2)\langle\chi_1^2\rangle\langle\chi_v\rangle - 2\epsilon k_c^2\langle\chi_1^2\chi_v\rangle + k_c^2\langle\chi_v\chi_1\rangle\langle\chi_1\rangle;$$
$$c = k_c^2\langle\chi_v\chi_1\rangle\langle\chi_v\rangle - \epsilon k_c^2\langle\chi_v^2\chi_1\rangle.$$

Before we discuss the new, modified resonance, let us first examine the finite-ϵ regularization of the original resonance at k_1 in the Taylor state limit. Near the original resonance, i.e., $k_c^2 - k_1^2 \sim o(\epsilon)$,

$$a = -\epsilon k_c^2\langle\chi_1^3\rangle + o(\epsilon);$$
$$b = k_c^2\langle\chi_v\chi_1\rangle\langle\chi_1\rangle - 2\epsilon k_c^2\langle\chi_1^2\chi_v\rangle + o(\epsilon);$$
$$c = k_c^2\langle\chi_v\chi_1\rangle\langle\chi_v\rangle - \epsilon k_c^2\langle\chi_v^2\chi_1\rangle.$$

The large amplitude root is

$$\alpha_1^{(l)} = -\frac{b}{a} + \frac{c}{b} + O(\epsilon) = \frac{\langle\chi_v\chi_1\rangle\langle\chi_1\rangle}{\langle\chi_1^3\rangle}\frac{1}{\epsilon} - \frac{2\langle\chi_1^2\chi_v\rangle}{\langle\chi_1^3\rangle} + \frac{\langle\chi_v\rangle}{\langle\chi_1\rangle} + O(\epsilon). \tag{7}$$

This is a regularized solution over the original resonance. It is interesting to note that the regularized solution at $k_c = k_1$ in the earlier model, Eq. (1), scales with $\epsilon^{-1/2}$ [11] instead of ϵ^{-1} in Eq. (7) for the new model, Eq. (3).

Unlike the model [e.g., Eq. (1)] considered in Ref. [11], a modified resonance persists at a k_c that is shifted from k_1 by order ϵ. In other words, a modified resonance appears in the neighborhood of $k_c^2 - k_1^2 \sim \epsilon$, where

$$a = (k_c^2 - k_1^2)\langle\chi_1^2\rangle\langle\chi_1\rangle - \epsilon k_c^2\langle\chi_1^3\rangle;$$
$$b = k_c^2\langle\chi_v\chi_1\rangle\langle\chi_1\rangle + O(\epsilon);$$
$$c = k_c^2\langle\chi_v\chi_1\rangle\langle\chi_v\rangle + O(\epsilon).$$

The modified resonance is due to a vanishing $a(k_c^r) = 0$ at a k_c^r that is ϵ dependent,

$$k_c^{r2} = k_1^2\frac{\langle\chi_1^2\rangle\langle\chi_1\rangle}{\langle\chi_1^2\rangle\langle\chi_1\rangle - \epsilon\langle\chi_1^3\rangle}. \tag{8}$$

In the small ϵ limit of our analysis, one finds that the shift is linearly proportional to ϵ,

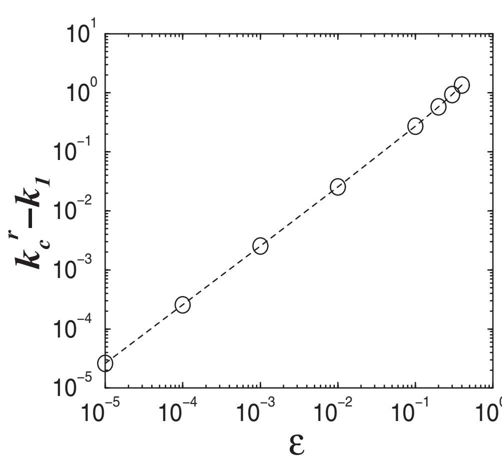

FIG. 3. The modified resonance $k_c^r(\epsilon)$ is upshifted from the original resonance k_1 by an amount approximately linearly proportional to ϵ in the first order model $k(\chi) = k_c(1 - \epsilon\chi/\chi_a)$.

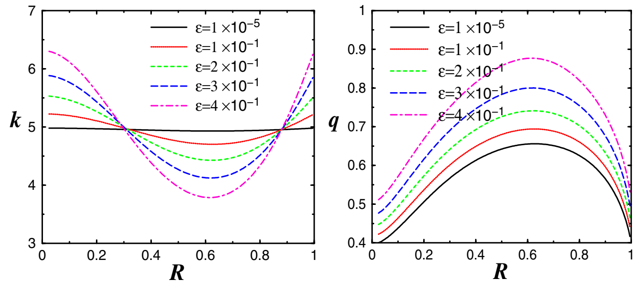

FIG. 4 (color online). The $k(\chi)$ (left) and $q(\chi)$ (right) profiles of the modified resonant modes for five ϵ's in the first order model are plotted as functions of the major radius R in the midplane.

$$k_c^r - k_1 = \epsilon k_1 \frac{\langle \chi_1^3 \rangle}{2\langle \chi_1^2 \rangle \langle \chi_1 \rangle}. \qquad (9)$$

This linear scaling with the deviation ϵ from the fully relaxed Taylor state is in agreement with the numerical computation; Fig. 3. Equation (8) also suggests that the shift in the nominal resonant frequency (k_c^r) depends on the Chandrasekhar-Kendall modes which are properties of the chamber geometry. The particular form of vacuum injector flux χ_v does not affect the modified resonant frequency k_c^r. This is a valid statement even if the analysis is carried out to arbitrary order in the expansion series for χ in terms of χ_v and χ_i's.

The actual modified resonant modes, which are ϵ dependent, have their unique $k(\chi(\mathbf{x}))$ profiles and hence $q(\chi(\mathbf{x}))$ profiles; Fig. 4. For the edge-peaked current profile ($\epsilon > 0$) under consideration, the resonant modes have about the same volume averaged value for k, although the more peaked profile (larger ϵ) tends to produce a higher q; Fig. 4. The net toroidal-poloidal flux ratio in the modified resonant modes ($\epsilon > 0$) is greater than that in the original Chandrasekhar-Kendall mode by an amount linearly proportional to ϵ,

$$\Delta(\Psi^t/\chi^t) \equiv \frac{\Psi^t(\epsilon)}{\chi^t(\epsilon)} - \frac{\Psi^t(\epsilon = 0)}{\chi^t(\epsilon = 0)} \propto \epsilon, \qquad (10)$$

as shown in Fig. 5.

We conclude that a modified driven resonance persists in a partially relaxed plasma as long as the deviation from the fully relaxed Taylor state is a function of the normalized magnetic flux χ/χ_a with χ_a the instantaneous poloidal flux at the magnetic axis. Since driven resonance is what gives rise to flux amplification in helicity injection experiments, an implication of practical importance is a potential path to achieve high field strength spheromak by high flux amplification and hence high current multiplication in a partially relaxed plasma that deviates significantly from the Taylor state. This new understanding of driven resonance thus motivates careful future experiments to understand the laboratory accessibility of the new driven resonance. In other words, does the laboratory plasma develop an incomplete relaxation that has a flux dependence normalized by

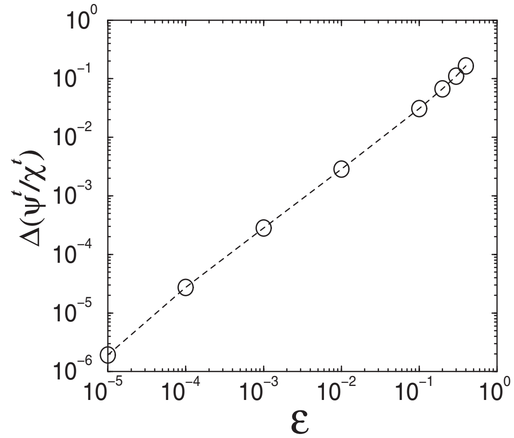

FIG. 5. The deviation of the toroidal to poloidal flux ratio of the modified resonant modes from that of the original Chandrasekhar-Kendall mode is plotted as a function of ϵ in the first order model.

the instantaneous flux, e.g., Eq. (1), or a preset one, e.g., Eq. (3), and under what condition? It is to be expected that even if the primary flux dependence is normalized by χ_a for which the modified driven resonance persists, there will likely be additional χ_a-independent nonlinearity that regularizes the modified resonance. The important point is that the additional χ_a-independent deviation, which hopefully is a small part of the total deviation from Taylor state, permits a much higher bound for spheromak flux amplification and hence current multiplication.

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