

Shear Viscosity in Weakly Coupled $N = 4$ Super Yang-Mills Theory Compared to QCD

Simon C. Huot, Sangyong Jeon, and Guy D. Moore

Physics Department, McGill University, 3600 rue University, Montréal, QC H3A 2T8, Canada

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We compute the shear viscosity of weakly coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Our result for η/s , the viscosity to entropy-density ratio, is many times smaller than the corresponding weak-coupling result in QCD. This suggests that η/s of QCD near the transition point is several times larger than the viscosity bound, $\eta/s \geq 1/4\pi$.

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A major surprise that has emerged from the experimental study of heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) is the large size of radial and elliptic flow observed in the collisions [1]. Indeed, the strength of the elliptic flow observed in noncentral heavy ion collisions is as large as in models where the plasma in the early stages after the collision is an ideal fluid [2]. (A fluid is nearly ideal if local equilibration proceeds on length and time scales small compared to the scale of inhomogeneity of the system.) This is a surprise because the plasma droplet created in heavy ion collisions is not much larger than its intrinsic microscopic length scale—roughly, the de Broglie wavelength of a typical excitation. A fluid's non-ideality is set by $(l_{\text{intrinsic}}/l_{\text{inhom}})(\eta/s)$, where η is the shear viscosity and s is the entropy density. (η/s is dimensionless in units with $\hbar = 1 = k_B$.) Therefore, η/s for the quark-gluon plasma must be numerically small to display nearly ideal fluid behavior; a value $\eta/s > 0.2$ may be enough to reduce the elliptic flow below what is observed [3] (though the data may actually require some nonideality [4]).

At sufficiently large energy densities, QCD is weakly coupled and η/s is relatively large, $\eta/s \sim 1/\alpha_s^2 \ln \alpha_s^{-1}$. However, the temperatures achieved at RHIC (probably below 0.5 GeV) are such that the (running) coupling is not small. Attempts to interpolate [5] between the behavior of η/s at high temperature [6] and low temperature [7] suggest $\eta/s \sim 1$ at the relevant temperatures. Unfortunately, the only first-principles technique we have in this strongly coupled region is lattice QCD, which suffers large uncertainties when extracting real-time behavior such as η [8].

Recently, a new theoretical perspective arose. There is a theory closely related to QCD, namely $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM), where calculations can be performed in the limit of a large number of colors N_c and strong 't Hooft coupling $\lambda \equiv g^2 N_c$ by using string theory techniques. (The 't Hooft coupling correctly accounts for the effective coupling strength given the large number of degrees of freedom involved.) The shear viscosity of SYM theory for large N_c and λ has been computed; expressed as the ratio η/s , it is [9]

$$\frac{\eta}{s} = \frac{1}{4\pi} \left(1 + \frac{135\zeta(3)}{8(2\lambda)^{3/2}} + \dots \right). \quad (1)$$

It has further been conjectured by Kovtun, Son, and Starinets, after evaluating η/s in several related strongly coupled theories, that $\eta/s = \frac{1}{4\pi}$ is in fact a lower bound on η/s in all systems [10]. Together with the formal similarities between SYM theory and QCD, and the expectation that QCD is strongly coupled at the temperatures relevant in heavy ion collisions, there is a belief that QCD nearly saturates this bound, and that strongly coupled SYM theory may resemble QCD near the transition [11] and may be useful for describing other properties of strongly coupled QCD [12–14]. This argument is also supported by thermodynamic information; lattice calculations of the pressure of QCD [15] show that (at a few times the transition temperature) the pressure is close to $\frac{3}{4}$ of the value at zero coupling, exactly the ratio obtained in strongly coupled SYM theory [16].

To explore whether QCD saturates the viscosity bound, we think it is useful to examine more carefully how much SYM theory really behaves like QCD. In particular, there is a regime where calculations can be carried out in both theories—weak coupling. How close are the values of η/s in SYM theory and in QCD at weak coupling? And is the physics which sets the viscosity in the two theories the same? This Letter will address this question.

At weak coupling, a gauge theory plasma behaves much like a gas of quasiparticles. A rough estimate of the viscosity and entropy density are

$$\eta \sim l_{\text{mfp}} \bar{v} (P + \varepsilon) \sim l_{\text{mfp}} n T, \quad s \sim n, \quad (2)$$

with n the number density of excitations and T the temperature. Hence, η/s is a measure of the ratio of the mean free path for large angle scattering, l_{mfp} , to the thermal length $1/T$. In practice, l_{mfp} is momentum dependent and scatterings do not fully randomize a particle's direction. To turn the estimate into a calculation, we must use kinetic theory (Boltzmann equations). The single particle distribution function $f(\mathbf{p}, \mathbf{x})$ evolves according to

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}}) f^a(\mathbf{p}, \mathbf{x}, t) = -C_a[f], \quad (3)$$

with $C_a[f]$ a collision term we explain below. Shear viscosity is relevant when the equilibrium distribution $f_0(\mathbf{p}, \mathbf{x}, t) = (\exp[\gamma(E + \mathbf{u} \cdot \mathbf{p})/T] \pm 1)^{-1}$ varies in space, $\partial_i u_j \neq 0$. The stress tensor (in the local fluid frame) will be

shifted by an amount proportional to this velocity gradient and η is the proportionality constant, $T_{ij} - T_{ij,\text{eq.}} = -\eta(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u_k) - \zeta\delta_{ij}\partial_k u_k$. (We will not discuss the bulk viscosity ζ .) The left-hand side of Eq. (3), at lowest order in $\partial_i u_j$, reads

$$(\partial_i u_j) f_0 (1 \pm f_0) \frac{p_i p_j}{ET} = -C_a[f], \quad (4)$$

while the relevant entry in the stress tensor is

$$T_{ij} - T_{ij,\text{eq.}} = \sum_a \int \frac{d^3 p}{(2\pi)^3} \frac{p_i p_j}{E} [f_a(\mathbf{p}, \mathbf{x}) - f_0(\mathbf{p}, \mathbf{x})]. \quad (5)$$

Both expressions involve the combination $p_i p_j / E$. The collision term is linear in the departure from equilibrium and must be inverted to solve for $f - f_0$.

Arnold, Moore, and Yaffe have shown how the inversion of the collision term can be performed in QCD by variational methods to determine the viscosity [6]. We follow their treatment, leaving the full details to the references. For each species, one must model the departure from equilibrium by a several parameter *ansatz*, with trial functions $\phi^{(m)}(p)$. To find the viscosity, one needs the integral of \mathbf{p}^2/E against each trial function, $S_{am} \equiv \nu_a \int_p (\mathbf{p}^2/E) \phi^{(m)}(p) f_0(1 \pm f_0)$ (ν_a the multiplicity of species a) and the integral moments of the collision operator, $C_{am,bn} = \int_p \nu_a \phi^{(m)}(p) C_a[f_b(k) = f_0 + f_0(1 \pm f_0)\phi^{(n)}(k)P_2(\cos\theta_{pk})]$, with $P_2(\cos\theta_{pk})$ the second Legendre polynomial of the angle between \mathbf{p} and \mathbf{k} . The viscosity is [6]

$$\eta = \frac{S_{am} C_{am,bn}^{-1} S_{bn}}{15T}, \quad (6)$$

where (am) is treated as a single index and C as a matrix. Their results in massless QCD, divided by the leading-order entropy density $s = 2\pi^2 T^3 g_*/45$ (with g_* the number of bosonic fields plus 7/8 the number of fermionic fields, $16 + 36(7/8) = 47.5$ for 3-flavor QCD), are

$$\frac{\eta}{s} \approx \frac{A}{N_c^2 g^4 \ln(B/g\sqrt{N_c})}, \quad A, B = \begin{cases} 34.8, 4.67 & N_f = 0, \\ 46.1, 4.17 & N_f = 3. \end{cases} \quad (7)$$

Massless QCD with $N_f = 3$ is the case closest to the real world, because the temperatures in heavy ion collisions have $m_u, m_d, m_s < T$, but $m_c \gg T$.

Now we apply their technique to $\mathcal{N} = 4$ SYM theory. This is a theory containing gauge fields for an $SU(N_c)$ symmetry, 4 adjoint Weyl fermions, and 6 real adjoint scalar fields, with Lagrangian density given in [17].

As in QCD, two types of collision processes are relevant; elastic $2 \leftrightarrow 2$ processes and inelastic effective $1 \leftrightarrow 2$ processes. The latter are really splitting/joining processes induced by soft scattering in the plasma.

The vacuum, squared $2 \leftrightarrow 2$ process matrix elements are displayed in Table I. The integral of these matrix elements over possible external momenta must be performed by

TABLE I. Matrix elements squared, summed on all external states of given spin, where S, F, G are spin 0, $\frac{1}{2}$, 1, respectively. Here $M_t = 2[\frac{u^2+s^2}{t^2} + 1]$, $X_{us} = 2[-\frac{u}{s} - \frac{s}{u} - 1]$, and M_u and M_s are M_t with $u \leftrightarrow t$ and $s \leftrightarrow t$, respectively. The remaining elements can be found by crossing.

$SS \rightarrow SS$	$36(M_t + M_s + M_u)$
$SG \rightarrow SG$	$12M_t$
$GG \rightarrow GG$	$4(M_t + M_s + M_u)$
$FS \rightarrow FS$	$48M_t + 144X_{us}$
$FS \rightarrow FG$	$48X_{us}$
$FG \rightarrow FG$	$16M_t + 16X_{us}$
$FF \rightarrow FF$	$64(M_t + M_s + M_u)$

numerical quadratures. Those elements containing M_t or M_u require bosonic, and X_{us} and X_{tu} require fermionic, self-energy corrections, which are the same as in QCD [6], but with $m_D^2 = 2\lambda T^2$ and $m_F^2 = \lambda T^2/2$.

The splitting processes are in a way simpler than in QCD; all particles are in the adjoint representation and all hard particles have the same dispersion relation: $E^2 = p^2 + \lambda T^2$ for $p^2 \gg \lambda T^2$. However, there are more possible splitting processes, because both gauge and Yukawa interactions can induce splitting. The total contribution to $C_{am,bn}$ due to splitting processes is

$$C_{am,bn} = 2 \sum_{ABC} \nu_{ABC} \frac{\lambda^2}{2(2\pi)^3} \int_0^\infty dp \int_{p/2}^p dk I, \quad (8)$$

$$I = \mathcal{J}_{ABC}(p, k) \mathcal{F}(p, k) e^{p/T} f_A(p) f_B(k) f_C(p-k) \\ \times [\delta_{aA} \phi_A^{(m)}(p) - \delta_{aB} \phi_B^{(m)}(k) - \delta_{aC} \phi_C^{(m)}(p-k)] \\ \times [\delta_{bA} \phi_A^{(n)}(p) - \delta_{bB} \phi_B^{(n)}(k) - \delta_{bC} \phi_C^{(n)}(p-k)]. \quad (9)$$

Here \mathcal{J} is the splitting kernel given in Table II and \mathcal{F} is the solution to the integral equation

$$\mathcal{F}(p, k) = \lambda^{-1} \int \frac{d^2 \mathbf{h}}{(2\pi)^2} 2\mathbf{h} \cdot \mathbf{F}(\mathbf{h}), \quad (10)$$

$$2\mathbf{h} - i\delta E \mathbf{F}(\mathbf{h}) = \int \frac{d^2 \mathbf{q}}{(2\pi)^2} \frac{\lambda^2 T^3}{\mathbf{q}^2(\mathbf{q}^2 + 2\lambda T^2)} \\ \times \left\{ \sum_{l=p, k, p-k} [\mathbf{F}(\mathbf{h} + l\mathbf{q}) - \mathbf{F}(\mathbf{h})] \right\} \\ \delta E = \frac{\mathbf{h}^2}{2pk(p-k)} + \frac{[p^2 + k^2 + (p-k)^2]\lambda T^2}{4pk(p-k)}. \quad (11)$$

The equation for \mathbf{F} accounts for splitting due to multiple scattering, with \mathbf{q} the transverse momentum exchange due to a single scattering and $\mathbf{h} \equiv \mathbf{p} \times \mathbf{k}$.

Using the procedure of Ref. [6] and this collision term, we find that the viscosity at next-to-leading-log order is

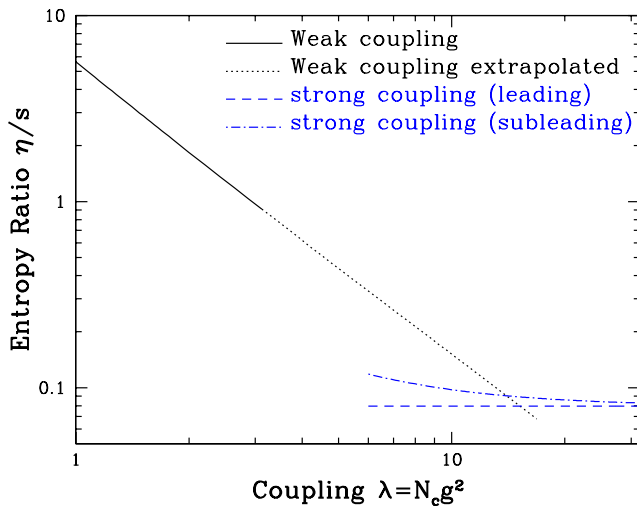
$$\frac{\eta_{\text{SYM}}}{s_{\text{SYM}}} \approx \frac{6.174}{\lambda^2 \ln(2.36/\sqrt{\lambda})}, \quad (12)$$

TABLE II. Splitting kernels for allowed 3-body processes in $\mathcal{N} = 4$ SYM theory; all others can be found by crossing.

ABC	ν_{ABC}	$\mathcal{J}_{ABC}(p, k)$
SFF	12	$p^2 k(p-k)/p^3 k^3 (p-k)^3$
GSS	3	$2k^2(p-k)^2/p^3 k^3 (p-k)^3$
GFF	4	$k(p-k)[k^2 + (p-k)^2]/p^3 k^3 (p-k)^3$
GGG	1	$[p^4 + k^4 + (p-k)^4]/p^3 k^3 (p-k)^3$

where as before, $\lambda \equiv N_c g^2$ is the 't Hooft coupling. At leading order, η/s is a complicated function of λ which must be determined numerically. We resolve the $O(\sqrt{\lambda})$ ambiguities in its determination using the procedure of Ref. [6]. Our (N_c independent) result is plotted in Fig. 1, which also shows the strong-coupling asymptotic. The dotted part of the weak-coupling curve is where we believe that corrections to the weak-coupling calculation may exceed the factor-of-2 level, so the curve guides the eye rather than being a firm calculation. (In the one theory where we have an all-orders calculation of η/s , namely, large N_f QCD [18], the leading-order and exact results deviate by about a factor of 2 when the Debye screening mass m_D reaches the same value as where we switch to a dotted line in Fig. 1.) Similarly, the large-coupling asymptotic cannot be trusted where it is not close to the large λ value of $1/4\pi$. The curves suggest that strong coupling behavior sets in around $\lambda \sim 15$. For comparison, the pressure of SYM theory appears to cross from weak to strong coupled behavior at a much smaller value of the 't Hooft coupling, $\lambda \sim 4$ [19].

Our result for weakly coupled SYM theory is much smaller than the result for QCD, with or without fermions, at the same coupling, as shown in Fig. 2. Naively, this suggests that the viscosity of QCD at strong coupling should be of order 7 times larger than that of SYM theory,

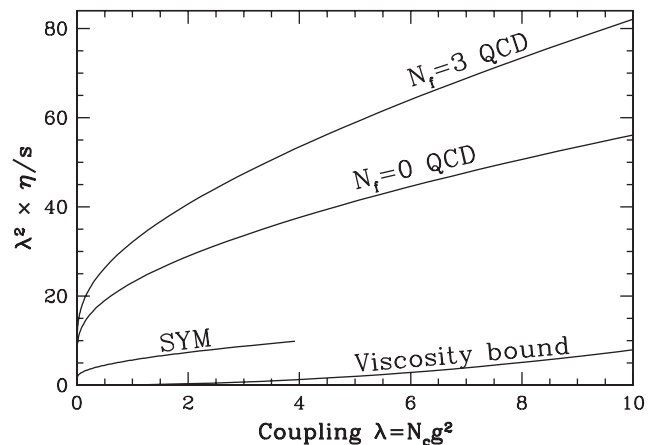
FIG. 1 (color online). Shear viscosity to entropy-density ratio η/s in $\mathcal{N} = 4$ SYM theory. The dotted curve is the weak-coupling calculation pushed beyond its likely range of validity.

far from the viscosity bound and closer to the values for other fluids near critical points. However, we should explore this conclusion a little more carefully, to try to understand how this large difference arose.

The main physics determining the shear viscosity at weak coupling is Coulomb scattering. Neglecting all scattering processes but Coulomb scattering changes, the leading-log coefficient A of Eq. (7) by less than 3% (0.2%) for $N_f = 3$ QCD (SYM theory). Working beyond logarithmic order, neglecting all processes but Coulomb scattering shifts our viscosity result by $O(25\%)$. Therefore, to good approximation the physics we must compare between theories is the physics of Coulomb scattering.

Two coupling strengths are relevant in Coulomb scattering; the coupling of a quasiparticle to gauge bosons, and the coupling of that gauge boson to all other degrees of freedom in the plasma. The first coupling (summed over available gauge bosons) goes as $C_R g^2$ with C_R the relevant group Casimir operator. In the case of SYM theory, $C_R = C_A = N_c$; for QCD it is $N_c = 3$ for gluons and $(N_c^2 - 1)/2N_c = 4/3$ for quarks. The second factor depends on the number, representation, and statistics of the other degrees of freedom in the plasma, in exactly the combination which enters in the Debye screening mass squared. Therefore it is natural to expect $s/\eta \sim C_R g^2 (m_D^2/T^2)$.

The quarks in SYM theory are adjoint rather than fundamental, leading to about a factor of 2 in the Casimir operator and $1/2$ in η/s . But much more importantly, the degree-of-freedom count which enters in m_D^2 is substantially larger in SYM theory than in QCD. For instance, for $N_c = 3$, SYM theory has $4 \times 2 \times 8 = 64$ fermionic degrees of freedom (four Weyl fermion species, consisting of a particle and antiparticle in 8 colors) and $(6 + 2) \times 8 = 64$ bosonic degrees of freedom (6 scalars and 2 gauge boson polarization states times 8 colors), for a total of 128; while in $N_f = 3$ QCD, there are only $3 \times 4 \times 3 = 36$ fermionic and 16 bosonic degrees of freedom. The

FIG. 2. η/s for SYM theory and for QCD, scaled by the dominant λ dependence and plotted as a function of λ . The value in SYM theory is dramatically smaller than in QCD.

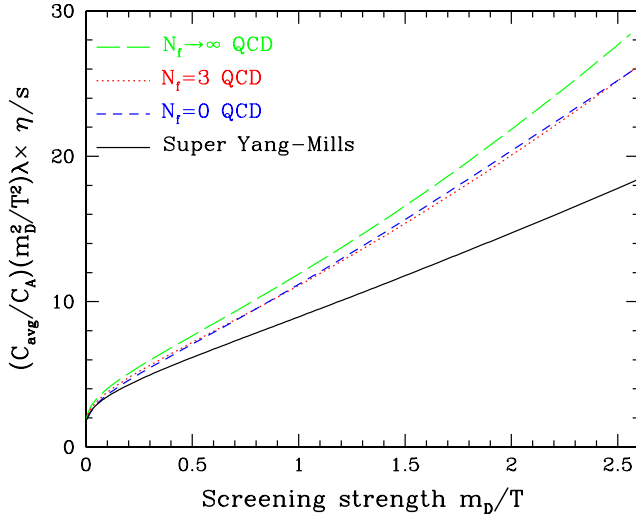


FIG. 3 (color online). Leading-order η/s for SYM theory and for QCD, scaled by $C_{\text{avg}} \lambda m_D^2 / T^2$, which we argue in the text captures the dominant coupling dependence, as a function of m_D / T .

Debye masses are, correspondingly, very different; $m_D^2 = 2\lambda T^2$ in SYM theory, but $m_D^2 = \frac{1}{2}\lambda T^2$ in $N_c = 3$, $N_f = 3$ QCD.

Since $\eta \propto l_{\text{mfip}}$, which scales as C_R^{-1} for each species, a reasonable way to account for the difference in the group Casimir operators in a theory like $N_f = 3$ QCD, with particles in multiple representations, is to take an average of the inverse Casimir operators. Therefore, define

$$C_{\text{avg}}^{-1} \equiv \frac{C_{\text{matt}}^{-1} g_{\text{matt}} + C_A^{-1} g_{\text{adj}}}{g_*}, \quad (13)$$

with C_{matt} the Casimir operator for the representation of matter fields and $g_{\text{matt}}, g_{\text{adj}}$ the contributions of each type of field to g_* defined earlier. Scaling η by $C_{\text{avg}} m_D^2 / C_A T^2$ greatly improves the agreement between different theories, as shown in Fig. 3. However, η/s in SYM theory remains below that in QCD, because SYM theory has Yukawa and scalar interactions which are absent in QCD and which introduce additional scattering channels, further lowering η . While these do not contribute at leading-log, they become more important as the coupling increases.

To conclude, weak-coupling comparisons of QCD with $\mathcal{N} = 4$ super Yang-Mills theory strongly suggest that QCD will not approach the viscosity bound, $\eta/s \geq 1/4\pi$, close to the QCD phase transition or crossover point. At weak coupling and equal values of t'Hooft coupling, SYM theory has a value of η/s which is about 1/7 that of QCD. This difference arises because quarks have a smaller coupling to gluons than the gluon self-coupling, since they are in a different group representation; and more significantly, because SYM theory has many more degrees of

freedom available as scattering targets than does QCD. A more reasonable comparison, both in terms of the size of η/s and the reliability of thermal perturbation theory, is to compare theories at the same value of m_D^2 / T^2 . Relating couplings in this way, $\alpha_s = 0.5$ corresponds to $\lambda = 4.7$, where SYM theory has thermodynamics very close to the strongly coupled behavior but η/s still about 6 times larger than the viscosity bound. It might be reasonable to believe that QCD plasmas relevant at RHIC fall in this region.

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