Cancellations Beyond Finiteness in $\mathcal{N} = 8$ Supergravity at Three Loops

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We construct the three-loop four-point amplitude of $\mathcal{N} = 8$ supergravity using the unitarity method. The amplitude is ultraviolet finite in four dimensions. Novel cancellations, not predicted by traditional superspace power-counting arguments, render its degree of divergence in D dimensions no worse than that of $\mathcal{N} = 4$ super-Yang-Mills theory—a finite theory in four dimensions. Similar cancellations can be identified at all loop orders in certain unitarity cuts, suggesting that $\mathcal{N} = 8$ supergravity may be a perturbatively finite theory of quantum gravity.

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While physicists do not yet know how to construct an ultraviolet-finite, pointlike quantum field theory of gravity in four dimensions, neither have they shown that such a construction is impossible. Pointlike theories of gravity are nonrenormalizable because the gravitational coupling is dimensionful. To date, no known symmetry has proven capable of taming the divergences, leading to the widespread belief that all such theories require new physics in the ultraviolet (UV). These beliefs were historically an important motivation for the development of string theory. Were a finite four-dimensional pointlike theory of gravity to be found, surely either a new symmetry or nontrivial dynamical mechanism must underpin it. The discovery of either would have a fundamental impact on our understanding of gravity.

Supersymmetry has been studied extensively as a mechanism for taming UV divergences (see, e.g., Refs. [1,2]). Although assumptions about the existence of different types of superspaces lead to different power counting, any superspace argument delays the onset of divergences only by a limited number of loops. For example, pure minimal ($\mathcal{N} = 1$) supergravity cannot diverge until at least three loops [3,4]. For maximal $\mathcal{N} = 8$ supergravity [5], were a fully covariant superspace to exist, divergences would be delayed until seven loops. With the additional (unconventional) assumption that all fields respect ten-dimensional general coordinate invariance, one can even delay the divergence to nine loops [6]. Recent arguments [7] using the type II string nonrenormalization theorems of Berkovits [8] suggest that divergences in the corresponding supergravity theory may indeed not arise before this loop order, though issues with smoothness in the low-energy limit do weaken this prediction [7]. Beyond this order, no known purely supersymmetric mechanism can avoid divergences. String dualities also hint at UV finiteness for $\mathcal{N} = 8$ supergravity [9], unless the situation is spoiled by towers of light nonperturbative states from branes wrapped on the compact dimensions [10].

Nonetheless, a different line of reasoning [11] using the unitarity method [12] has provided direct evidence that $\mathcal{N} = 8$ supergravity may be UV finite to *all* loop orders [13]. (See also Ref. [14].) At one loop, all known multigraviton amplitudes in the theory (including all with up to six gravitons) can be expressed solely in terms of scalar box integrals; neither triangle nor bubble integrals appear [15,16]. Supersymmetry, factorization, and infrared arguments provide strong evidence that the same is true for all one-loop amplitudes. This "no-triangle hypothesis" [16] implies a set of surprising cancellations which go beyond any known superspace argumentation. Generalized unitarity cuts, isolating one-loop subamplitudes inside higher-loop amplitudes, then imply specific multiloop cancellations [13]. Are similar cancellations present in all contributions to multiloop amplitudes, and do they render the theory UV finite?

In this Letter, we take a concrete step toward addressing these questions by presenting the complete three-loop four-point amplitude of $\mathcal{N} = 8$ supergravity. Details of the computation will appear elsewhere [17]. Here we show that the amplitude possesses the cancellations expected if the theory were indeed finite to all loop orders.

Reference [11] analyzed iterated two-particle cuts to all loop orders and argued that $\mathcal{N} = 8$ supergravity is finite for

$$D < 10/L + 2 \qquad (L > 1), \tag{1}$$

where L is the loop order and D is the dimension. (For $L = 1$, the finiteness bound is $D < 8$, not $D < 12$.)

A similar analysis for $\mathcal{N} = 4$ super-Yang-Mills theory [11,18] gives the finiteness condition,

$$D < 6/L + 4 \qquad (L > 1). \tag{2}$$

The bound (2) differs somewhat from earlier superspace power counting [19] although all bounds confirm UV finiteness of $\mathcal{N} = 4$ super-Yang-Mills theory in $D = 4$. The bound (2) has been proven to all orders using $\mathcal{N} = 3$

harmonic superspace [2]. Explicit computations show that it is saturated through four loops [11,13,18,20].

The $\mathcal{N} = 8$ supergravity bound (1) corresponds, in the language of effective actions, to a one-particle irreducible effective action starting with loop integrals multiplied by $\mathcal{D}^4 R^4$ at each loop order beyond $L = 1$. Here R^4 is a shorthand for the supersymmetrization of a particular contraction of four Riemann tensors [4], and $\mathcal{D}$ denotes a generic covariant derivative. The stronger bound (2), if applied to $\mathcal{N} = 8$ supergravity, would differ from Eq. (1) beginning at $L = 3$ for general D, although both bounds imply three-loop finiteness for $D = 4$. It corresponds to a three-loop effective action beginning with $\mathcal{D}^6 R^4$, not $\mathcal{D}^4 R^4$. As the supergravity finiteness bound (1) is based on only a limited set of unitarity cuts [11], additional (stronger) cancellations may be missed [13].

To study this issue, we use the unitarity method [12,18] to build the three-loop four-point $\mathcal{N} = 8$ supergravity amplitude. In this method, on-shell tree amplitudes suffice as ingredients for computing amplitudes at any loop order. The reduction to tree amplitudes is crucial. It allows the use of the Kawai-Lewellen-Tye (KLT) [21] tree-level relations between gravity and gauge-theory amplitudes [11], effectively reducing gravity computations to gauge-theory ones. The original KLT relations express tree-level closed-string scattering amplitudes in terms of pairs of open-string ones. The perturbative massless states of the closed and open type II superstring compactified to four dimensions on a torus are those of $\mathcal{N} = 8$ supergravity and $\mathcal{N} = 4$ super-Yang-Mills theory, respectively. Thus, in the limit of energies well below the string scale, the KLT relations express $\mathcal{N} = 8$ supergravity tree amplitudes as quadratic combinations of $\mathcal{N} = 4$ super-Yang-Mills tree amplitudes (see, e.g., Ref. [15]). At tree level, there are no subtleties in taking this limit.

We use the generalized unitarity cuts [22] illustrated in Fig. 1. Together with the iterated two-particle cuts evaluated in Refs. [11,18], these cuts completely determine any massless three-loop four-point amplitude. Since we are interested in the UV behavior of the amplitudes in D dimensions, the unitarity cuts must be evaluated in D dimensions [23]. This renders the calculation more difficult because powerful four-dimensional spinor methods cannot be used. Some of the D-dimensional complexity is avoided by performing internal-state sums in terms of the simpler on-shell gauge supermultiplet of $D = 10$, $\mathcal{N} = 1$ super-Yang-Mills theory instead of the $D = 4$, $\mathcal{N} = 4$ multiplet. We have also performed various four-dimensional cuts, which in practice provide a very useful guide.

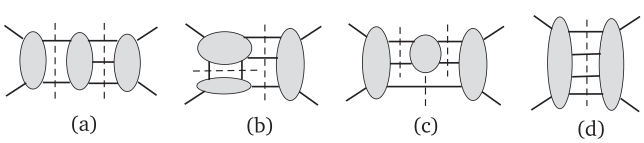

FIG. 1. Generalized cuts used to determine the three-loop four-point amplitude.

Our computation proceeds in two stages. In the first stage, we deduce the three-loop $\mathcal{N} = 4$ super-Yang-Mills amplitudes from generalized cuts, including cuts (a)–(c) in Fig. 1, and the iterated two-particle cuts analyzed in Refs. [11,18]. From the cuts, we obtain a loop-integral representation of the amplitude. The diagrams in Fig. 2 describe the scalar propagators for the loop integrals. The numerator factor for each integral in the super-Yang-Mills case is given in the second column of Table I.

In the second stage, we use the KLT relations to write the cuts of the $\mathcal{N} = 8$ supergravity amplitude as sums over products of pairs of cuts of the corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitude, including twisted nonplanar contributions. The iterated two-particle cuts studied in Ref. [11], together with the cuts in Fig. 1 evaluated here, suffice to fully reconstruct the supergravity amplitude. We find that the three-loop four-point $\mathcal{N} = 8$ supergravity amplitude in D dimensions is

$$M_4^{(3)} = \left(\frac{\kappa}{2}\right)^8 stu M_4^{\rm tree} \sum_{S_3} \bigg[I^{(a)} + I^{(b)} + \frac{1}{2} I^{(c)} + \frac{1}{4} I^{(d)} + 2I^{(e)} + 2I^{(f)} + 4I^{(g)} + \frac{1}{2} I^{(h)} + 2I^{(i)} \bigg], \qquad (3)$$

where S_3 represents the six independent permutations of legs $\{1, 2, 3\}$, κ is the gravitational coupling, and $M_4^{\rm tree}$ is the supergravity four-point tree amplitude. The $I^{(x)}(s, t)$ are D-dimensional loop integrals corresponding to the nine diagrams in Fig. 2, with numerator factors given in the third column of Table I. The Mandelstam invariants are $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_1 + k_3)^2$. The numerical coefficients in front of each integral in Eq. (3) are symmetry factors of the diagrams. Remarkably, the number of dimensions appears explicitly only in the loop integration measure.

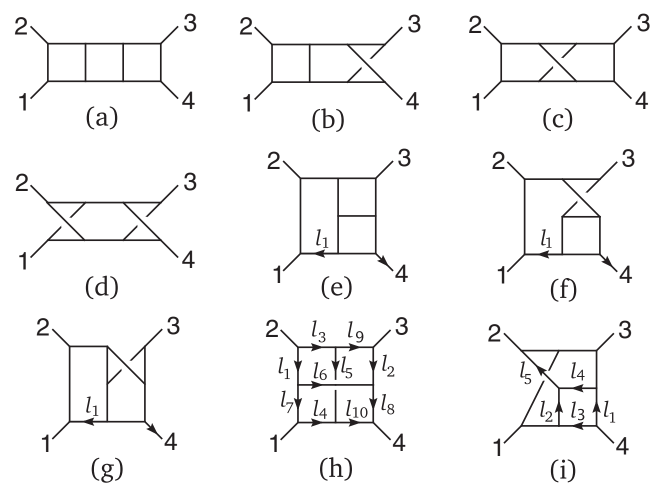

FIG. 2. Loop integrals appearing in both $\mathcal{N} = 4$ gauge-theory and $\mathcal{N} = 8$ supergravity three-loop four-point amplitudes. The integrals are specified by combining the diagrams' propagators with numerator factors given in Table I.

TABLE I. The numerator factors of the integrals $I^{(x)}$ in Fig. 2. The first column labels the integral, the second column the relative numerator factor for $\mathcal{N} = 4$ super-Yang-Mills, the third column the factor for $\mathcal{N} = 8$ supergravity. In the Yang-Mills case, an overall factor of $stA_4^{\rm tree}$ has been removed, while in the supergravity case, an overall factor of $stuM_4^{\rm tree}$ has been removed. ($A_4^{\rm tree}$ denotes a tree-level four-point amplitude of $\mathcal{N} = 4$ Super-Yang-Mills theory.) The loop momenta l_i are the momenta of the labeled propagators in Fig. 2, and $l_{i,j}^2 = (l_i + l_j)^2$.

Integral $I^{(x)}$	$\mathcal{N} = 4$ Super-Yang-Mills	$\mathcal{N} = 8$ Supergravity
(a)– (d)	s^2	$[s^2]^2$
(e)– (g)	$s(l_1 + k_4)^2$	$[s(l_1 + k_4)^2]^2$
(h)	$sl_{1,2}^2 + tl_{3,4}^2 - sl_5^2 - tl_6^2 - st$	$(sl_{1,2}^2 + tl_{3,4}^2 - st)^2 - s^2(2(l_{1,2}^2 - t) + l_5^2)l_5^2 - t^2(2(l_{3,4}^2 - s) + l_6^2)l_6^2$ $- s^2(2l_1^2 l_8^2 + 2l_2^2 l_7^2 + l_1^2 l_7^2 + l_2^2 l_8^2) - t^2(2l_3^2 l_{10}^2 + 2l_4^2 l_9^2 + l_3^2 l_9^2 + l_4^2 l_{10}^2) + 2stl_5^2 l_6^2$
(i)	$sl_{1,2}^2 - tl_{3,4}^2 - \frac{1}{3}(s - t)l_5^2$	$(sl_{1,2}^2 - tl_{3,4}^2)^2 - (s^2 l_{1,2}^2 + t^2 l_{3,4}^2 + \frac{1}{3}stu)l_5^2$

We remark that the amplitude (3) could be used to study $D = 11$, $\mathcal{N} = 1$ supergravity compactified on a circle or two-torus at three loops, just as the two-loop amplitude [11] was analyzed in Ref. [24]. That analysis, along with the assumption that M-theory dualities hold at this loop order, restricts the type II string effective action at three loops to start with $\mathcal{D}^6 R^4$, not $\mathcal{D}^4 R^4$.

With the explicit expression for the amplitude (3) in hand, we may determine the UV behavior straightforwardly. In the super-Yang-Mills case, the entries in the second column in Table I contain no more than two powers of loop momenta. Accounting for the ten propagators of each diagram in Fig. 2 and the three-loop integration measure, we see that each integral separately satisfies the known super-Yang-Mills finiteness bound (2).

In contrast, the supergravity numerators, as given in the third column of Table I, contain up to four powers of loop momenta. Separately, these integrals satisfy the bound (1). The iterated two-particle cuts evaluated in Ref. [11] give the integrals (a)–(g) in Fig. 2 and Table I. All such contributions have numerator factors which are squares of the Yang-Mills ones. The entries (h) and (i) are new and do not have this structure.

Might there be additional cancellations between the integrals in the $\mathcal{N} = 8$ supergravity case? To check this, we expand each integral in a series in the external momenta, keeping only the leading terms. We thereby keep terms with maximal powers of loop momenta and set the external momenta to zero in the propagators. Each integral reduces to a sum of vacuum diagrams, possibly with squared propagators. Figure 3 shows the resulting vacuum diagrams $V^{(x)}$. Integrals (a)–(d) in Fig. 2 have no powers of loop momenta in their numerators, and hence do not contribute to the leading UV behavior. The remaining integrals in Eq. (3) reduce as follows,

$$2I^{(\rm e)} \to 4V^{(\rm a)}, \qquad 2I^{(\rm f)} \to 4V^{(\rm b)}, \qquad 4I^{(\rm g)} \to 8V^{(\rm a)},$$
$$\frac{1}{2}I^{(\rm h)} \to -4V^{(\rm a)} - 8V^{(\rm b)} - 4V^{(\rm c)} - 2V^{(\rm e)}, \qquad (4)$$
$$2I^{(\rm i)} \to -8V^{(\rm a)} + 4V^{(\rm b)} + 8V^{(\rm d)},$$

taking into account the permutation sum over external legs and suppressing an overall factor of $(s^2 + t^2 + u^2)stuM_4^{\rm tree}$. Using a momentum-conservation identity,

$$V^{(\rm c)} = 2V^{(\rm d)} - \frac{1}{2}V^{(\rm e)}, \qquad (5)$$

to eliminate $V^{(\rm c)}$, *the coefficients of the remaining four vacuum diagrams cancel completely.*

Lorentz covariance implies that contributions with three powers of loop momenta in the numerator are no more divergent than integrals with only two powers. We have also found a rearrangement of the loop-momentum integrands which makes manifest this quadratic behavior, equivalent to the amplitude behaving as $\mathcal{D}^6 R^4$ [17]. Thus, the $\mathcal{N} = 8$ supergravity amplitude satisfies the same finiteness bound (2) at $L = 3$ as the corresponding $\mathcal{N} = 4$ super-Yang-Mills amplitude.

Some of these cancellations have an all-loop generalization that can be understood as a direct consequence of the no-triangle hypothesis [13]. Using generalized unitarity we may isolate all one-loop subamplitudes of an L-loop amplitude as shown in Fig. 4(a). For example, the cut shown in Fig. 4(b) rules out the appearance of the vacuum diagram $V^{(\rm a)}$, because it would imply the appearance of a triangle integral at one loop. Similarly, we can infer a cancellation between vacuum diagrams $V^{(\rm c)}$ and $V^{(\rm d)}$. (A squared propagator counts as two sides of a triangle integral.) At higher loops, the generalized cut in Fig. 4(a), together with the no-triangle hypothesis, implies that any leading-singularity vacuum diagram containing a triangle

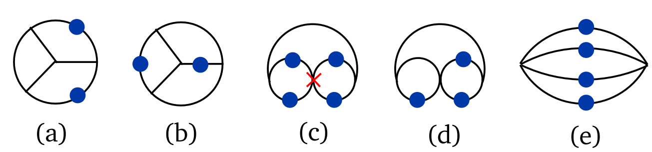

FIG. 3 (color online). The vacuum diagrams $V^{(x)}$ encoding the leading UV behavior of the individual $\mathcal{N} = 8$ supergravity diagrams. Dots on propagators represent squared propagators. The shaded cross in diagram (c) represents a numerator factor of l^2, where l is the momentum of a collapsed vertical propagator.

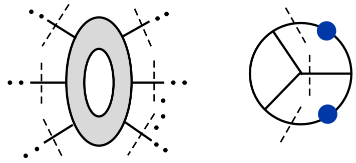

FIG. 4 (color online). A generalized cut (a) isolating a one-loop subamplitude in an L-loop amplitude. If a leg is external to the entire amplitude, it should not be cut. From the generalized cut (b), we see that diagram (a) in Fig. 3 must cancel since it has a one-loop triangle subdiagram.

subdiagram must have a vanishing coefficient. However, this argument does not suffice to rule out vacuum diagrams $V^{(b)}$ and $V^{(e)}$, because they have no triangle subdiagrams. Their coefficients nonetheless vanish, demonstrating the existence of cancellations *beyond* those implied by the no-triangle hypothesis.

This Letter establishes through three loops that the four-point amplitudes of $\mathcal{N} = 8$ supergravity have the same ultraviolet critical dimension (2) as the corresponding $\mathcal{N} = 4$ gauge-theory ones. Fourteen powers of loop momentum are extracted from the numerators of the three-loop integrals. This result is consistent with the manifest symmetries of an off-shell $\mathcal{N} = 7$ harmonic superspace [2], whose existence would imply UV finiteness for $D < 12/L + 2$. However, the cancellations we find go beyond this: generalized unitarity will propagate the additional three-loop cancellations, as well as the one-loop no-triangle constraint, into novel cancellations eliminating increasing powers of loop momenta at *all* loop orders [17].

To unravel the origin of these cancellations, and to constrain potential superspace explanations, it is important to compute additional $\mathcal{N} = 8$ amplitudes. Using the unitarity method, it should be feasible to compute the four- and five-loop four-point amplitudes, as well as the two-loop five-point amplitude. It should also be possible to carry out refined all-order studies, given the recursive nature of the formalism. In particular, it is important to investigate the classes of contributions not directly constrained by generalized unitarity and the no-triangle hypothesis.

The result presented here, in conjunction with the all-loop-order evidence from unitarity [13] and string theory hints of additional cancellations [7–9], points strongly towards the ultraviolet finiteness of $\mathcal{N} = 8$ supergravity.

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