Topological Computation without Braiding

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We show that universal quantum computation can be performed within the ground state of a topologically ordered quantum system, which is a naturally protected quantum memory. In particular, we show how this can be achieved using brane-net condensates in 3-colexes. The universal set of gates is implemented without selective addressing of physical qubits and, being fully topologically protected, it does not rely on quasiparticle excitations or their braiding.

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Topological quantum computation offers the possibility of implementing a fault-tolerant quantum computer avoiding the extremely low threshold error rates found with the standard quantum circuit model [1–4]. Physical systems exhibiting a topological quantum ordered state [5,6] can be used as naturally protected quantum memories [1,7,8]. Characteristic properties of topologically ordered systems are the energy gap between ground state and excited states, topology-dependent ground state degeneracies, braiding statistics of quasiparticles, edge states, etc. [6]. The idea is then to place the information in the topologically degenerate ground state of such a system, so that the protection of the encoded information comes from the gap and the fact that local perturbations cannot couple ground states. In fact, the probability of tunneling between orthogonal ground states is exponentially suppressed by the system size and vanishes in the thermodynamic limit.

A stabilizer code [9,10] can be topological. The best known example are Kitaev's surface codes [1,7]. In general a code is topological if its stabilizer has local generators and nondetectable errors are topologically nontrivial (in the particular space where the qubits are to be placed). Given such a code, one can always construct a local Hamiltonian such that the resulting system is topologically ordered and the error correcting code corresponds to the ground state. An explicit example of this Hamiltonian construction is given later in Eq. (3). Errors in the code amount to excitations.

Although the storage of quantum information is interesting by itself, one would like to perform computations on it. A natural approach in this context is that of considering a topological stabilizer code in which certain operators can be implemented transversally, which avoids error propagation within codes. In terms of the corresponding topologically ordered system, this means that operations are implemented without selective addressing of the physical subsystems that make up each memory. This is important for physical applications.

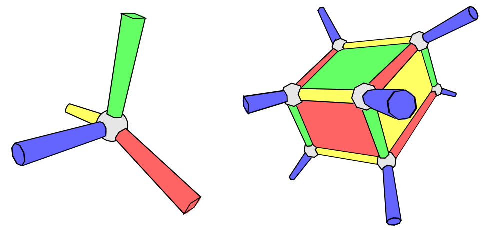
Unfortunately, surface codes only allow the transversal implementation of the CNOT gate. Then the problem arises of whether there exists a topological stabilizer code in which a universal set of gates can be performed transversally. In fact, even at the level of general codes it is a difficult task to find such codes [11]. For most codes, additional tricks such as the generation of large cat states are unavoidable. However, quantum Reed-Muller codes [12] have the very special property of allowing the transversal implementation of the gates:

$$K^{1/2} = \begin{pmatrix} 1 & 0 \\ 0 & i^{1/2} \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}, \tag{1}$$

where X is the usual σ_1 Pauli matrix. Complemented with the ability to initialize eigenstates of X and Z and to measure these operators, these gates are enough to perform arbitrary computations. In particular, the Hadamard gate can be reconstructed and the set of gates $\{H, K^{1/2}, \Lambda\}$ is known to be universal [13].

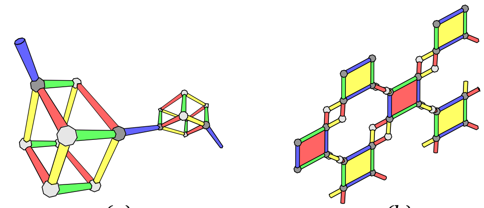
In this Letter we will construct a 3-dimensional system showing topological quantum order in which the gates (1) can be implemented. The ground state of the system is a topological stabilizer code. No other topological code of any dimension is known such that the transversal implementation of a universal set of gates is possible. In fact, a key ingredient in our approach is the appearance of membranes [14]. Our system is a 3-dimensional lattice with qubits located at the sites, and the operations on the ground state are implemented without any selective addressing of these physical qubits. This is in contrast with the current approach to topological computation which relies on the topological properties of quasiparticle excitations instead of ground states properties and needs a selective braiding of quasiparticles to produce quantum gates. In fact our system is Abelian, in the sense that monodromy operations on excitations can give rise only to global phases. In contrast, in the context of quasiparticle braiding Abelian systems can never give universal computation. Therefore, our results enlarge the range of applicability of the topological approach to quantum computation [15].

To achieve this goal, we start with a brief description of the topologically ordered 3-dimensional condensed matter systems [16] that we need for our construction. Consider a lattice with coordination number 4 in which links are colored with four colors as in Fig. 1(a). Color is introduced as a bookkeeping tool to keep track of the different sites,



(a) *(b)*

FIG. 1 (color online). (clockwise and lightest to darkest: yellow, green, red, blue) (a) A generic site in a 3-colex. (b) The neighborhood of a particular b cell: faces are colored according to the color of their visible side (they are br, bg, and by faces).



(a) *(b)*

FIG. 2 (color online). (a) A b string consists of several b links that connect b cells. (b) An ry membrane is made up of ry faces linked by bg faces. bg faces are not shown here, only their links.

links, faces, and cells in the 3D lattice. We will use red, green, blue, and yellow labels (r, g, b, y) as colors. Assume that the cells can also be colored, in such a way that, for example, the boundary links of a red cell is a net with coordination number 3 formed by green, blue, and yellow links, as in Fig. 1(b). We call those 3D lattices with this set of properties 3-colexes. For any color q, q links connect q cells. A face lying between an r and a y cell has a boundary link made up of g and b links. We call such a face an ry face.

At each site of the lattice we place a qubit. We will be considering operators of the form

$$B_S^\sigma := \bigotimes_{i=1}^{n} \sigma^{f_i}, \qquad \sigma = X, Z, \qquad f_i = \begin{cases} 0 & i \notin S, \\ 1 & i \in S \end{cases}, \tag{2}$$

where S is a given set of qubits in the system, n the total number of qubits. The Hamiltonian proposed in [16] is

$$H = -\sum_{c\in C} B_c^X - \sum_{f\in F} B_f^Z, \tag{3}$$

where C and F are the cells and faces of the lattice, respectively. It gives rise to topological order. In particular, the degeneracy of the ground state is 2^k with $k = 3h_1$, where h_1 is the number of independent cycles of the 3-manifold in which the lattice is built. In particular, in a 3-sphere $h_1 = 0$ and there is no degeneracy at all, whereas in a 3-torus $h_1 = 3$. In topology, h_1 is known as a Betti number [17].

The ground states $|\psi\rangle$ of (3) are characterized by the conditions

$$\forall\ c \in C \qquad B_c^X|\psi\rangle = |\psi\rangle, \tag{4}$$

$$\forall\ f \in F \qquad B_f^Z|\psi\rangle = |\psi\rangle. \tag{5}$$

In fact, cell and face operators commute, and the ground state is a stabilizer quantum error correcting code [1,9,18]. Those eigenstates $|\psi'\rangle$ for which any of the conditions (4) and (5) are violated are excited states or, in code terms, erroneous states.

Both excitations and degeneracy are best understood introducing string and membrane operators. A q string, for some color $q \in \{$r, g, b, y$\}$, is a collection of q links, as in Fig. 2(a). Strings can have end points, which are always located at q cells. Along with every q string s we introduce the string operator B_s^Z. If $|\psi\rangle$ is a ground state, then $B_s^Z|\psi\rangle$ is, in general, an excited state, with excitations or quasiparticles at those q cells that are end points of s. If s is closed, that is, if it has no end points, B_s^Z commutes with the Hamiltonian (3).

Similarly, a collection of pq faces, for any colors p and q, is a pq membrane, as in Fig. 2(b). For any pq membrane m the corresponding membrane operator is B_m^X. If $|\psi\rangle$ is a ground state and m an rg membrane, for example, then $B_m^X|\psi\rangle$ is in general an excited state, with excitations at those by faces that form the border of m. These excitations are closed fluxes crossing the excited faces. As an example, consider an ry membrane such as the one in Fig. 2(b). Its border will create an ry flux, which will cross those bg faces at the border of the membrane. If m is closed, that is if it has no borders, then B_m^X commutes with the Hamiltonian (3).

As long as we consider closed manifolds in 3D, closed strings and membranes are enough to form a basis from which any operator that leaves the ground state invariant can be constructed. There are three key points here. First, any two string or membrane operators which are equal up to a deformation have the same action on the ground state, which is in itself a uniform superposition generated by all the possible local deformations. Second, a q-string operator B_s^Z and a pq-membrane operator B_m^X anticommute if and only if s crosses m an odd number of times. Otherwise, they commute and the same is true if they do not share any color. Third, not all colors are independent. For example, the combination of an r, a g, and a b string gives a y string. In fact , there are exactly 3 independent colors for strings and 3 independent color combinations for membranes. Therefore, all that matters about strings and membranes is their color and topology, and the appearance of the number 3 in the degeneracy is directly related to the number of independent colors.

On the other hand, strings and membranes with a single color are not enough to describe excitations. In general, strings can form a net with branch points at which four strings meet, one for each color [see Figs. 3(a) and 4(b)] Likewise, membranes can form nets in which, for example, a gb, a br, and an rg membrane meet along a line [see Fig. 4(c)]. In order to study the exact properties of general excitations, one can consider the elementary excitations

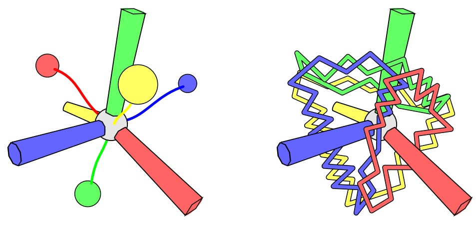


FIG. 3 (color online). (a) The Z operator of a site creates one quasiparticle at each of the cells that meet at the site. (b) The X operator of a site creates the flux structure shown, which corresponds to a flux excitation at each of the faces meeting at the site.

attached to the operators X and Z at any particular site i of the lattice. Let $|\psi\rangle$ be a ground state. Then the state $Z_i|\psi\rangle$ is an excited state with four quasiparticles; see Fig. 3(a). The state $X_i|\psi\rangle$ is an excited state with six elementary fluxes which can be arranged in four single color closed fluxes, as in Fig. 3(b). From this class of elementary excitations one can build any general excitation.

If we restrict ourselves to closed manifolds, there is no way in which we can have a ground state with twofold degeneracy, or equivalently, that encodes a single qubit. However, we will now explain how one can obtain such a system by puncturing a 3-manifold. In particular, consider any 3-colex in a 3-sphere. The ground state in this case is nondegenerate. Now we choose any site in the lattice and remove it. Moreover, we also remove the four links, six faces, and four cells that meet at the site. As a result, we obtain a lattice with the topology of a solid 2-sphere; see Fig. 4(a). In order to calculate the degeneracy of the new system, we note that we have removed one physical qubit and two independent generators [16] of the stabilizer. This is so because (i) although we remove 4 cells, three of the cell operators can be obtained from the remaining one and the rest of cell operators (see [16]), and (ii) although we remove 6 faces, 5 of the face operators can be obtained from the remaining one and the rest of face operators in the corresponding cells. Then, from the theory of stabilizer codes it readily follows that the new code encodes one qubit. This can also be understood using strings and membranes. The surface of the system is divided into four faces, each of them being the boundary with one of the removed cells. Thus, we can color these areas with each color of the faces from the removed cells, as in Fig. 4(a). It is natural to deform this sphere into a tetrahedron, and we will do so. Then each of its faces can be the end point of a string of the same color, and thus there is a single independent nontrivial configuration for a string-net, as depicted in Fig. 4(b). This configuration, of course, corresponds to a string-net operator that creates one quasiparticle excitation at each missing cell. In a similar fashion, one can consider a net of membranes that creates the flux configuration shown in Fig. 4(c). This net consists of six membranes, meeting in groups of three at four lines that

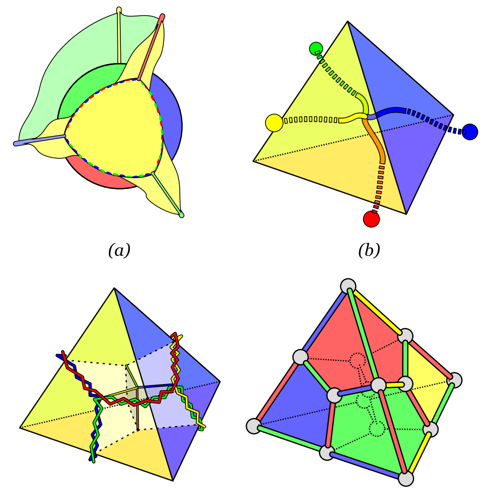


FIG. 4 (color online). (a) Here we represent the 3-sphere as $\mathbf{R}^3$ plus the point at infinity where we place the site to be removed. The faces and links perpendicular to the colored sphere are partially displayed but they continue to infinity. These faces and links must be removed as well. After their removal, we get a solid 2-sphere with a surface divided in four triangular areas. This colored sphere represents the remaining 3-colex itself. Then it can be reshaped to get a tetrahedron. (b) A nontrivial string-net in the tetrahedron. Its end points lie on the missing cells. (c) A nontrivial membrane-net configuration in the tetrahedron. Its borders create fluxes that cross the missing faces. Branching lines have been suitably colored. (d) The simplest tetrahedral lattice. Here colors have been given both to links and to cells. In the language of error correction, it is a [[15, 1, 5]] code, that is, it encodes a qubit in 15 physical qubits, whereas its distance is 5 and so corrects up to 2 errors.

meet at a central point. Observe that these excitations are in exact correspondence with those in Fig. 3, when we see them from the point of view of the removed site.

Although these string-net and membrane-net operators just discussed can be used to introduce an operator basis for the encoded qubit, this can be done in an alternative way that is more convenient for practical implementations. Given any operator O that acts on a single qubit, we will use the notation $\hat{O} := O^{\otimes n}$ for the operator that applies O to each of the n physical qubits in the 3D lattice. Then, in any tetrahedral lattice we have $\{\hat{X}, \hat{Z}\} = 0$, because the total number of sites is odd: every 3-colex has an even number of sites and we have removed one [see Fig. 4(d) for $n = 15$]. Since both $\hat{X}$ and $\hat{Z}$ commute with the Hamiltonian (4), they can be considered the X and Z Pauli operators on the protected qubit. As usual, let $|0\rangle$ and $|1\rangle$ be a positive and a negative eigenvector of Z, respectively, so that they form an orthogonal basis for the qubit state space. Let also $|\mathbf{v}\rangle := |v_1\rangle \otimes \cdots \otimes |v_n\rangle$ be a vector state for any binary vector $\mathbf{v} \in \mathbf{Z}_2^n$, $\mathbf{Z}_2 = \{0, 1\}$. These binary vectors are usually introduced in error correcting codes [10]. A basis for the protected qubit can be constructed as follows:

$$|\hat{0}\rangle := \prod_c (1 + B_c^X)|\mathbf{0}\rangle = \sum_{\mathbf{v}\in V} |\mathbf{v}\rangle, \tag{6}$$

$$|\hat{1}\rangle := \prod_c (1 + B_c^X)|\mathbf{1}\rangle = \hat{X}|\hat{0}\rangle = \sum_{\mathbf{v}\in V} |\bar{\mathbf{v}}\rangle, \tag{7}$$

where $\mathbf{0} := (0\cdots 0)$, $\mathbf{1} := (1\cdots 1)$, $\bar{\mathbf{v}} := \mathbf{1} + \mathbf{v}$, c runs over all cells in the lattice, and V is the subspace spanned by vectors $\mathbf{v}_c$ such that $|\mathbf{v}_c\rangle = B_c^X|\mathbf{0}\rangle$. In order to be able to apply the $K^{1/2}$ gate to the protected qubit in the tetrahedral lattice, we must introduce a new requirement. We impose that faces (cells) must have a number of sites which is a multiple of four (eight). The simplest example of such a tetrahedral lattice is displayed in Fig. 4(d). As we will show below, it follows from these conditions that

$$\forall\ \mathbf{v} \in V \quad \mathrm{wt}(\mathbf{v}) \equiv 0 \mod 8, \tag{8}$$

where the weight of a vector wt($\mathbf{v}$) is the number of 1's it contains. But then we have

$$\hat{K}^{1/2}|\hat{0}\rangle = |\hat{0}\rangle, \qquad \hat{K}^{1/2}|\hat{1}\rangle = i^{l/2}|\hat{1}\rangle, \tag{9}$$

where $l \equiv n \mod 8$, $l \in \{1, 3, 5, 7\}$. This means that the global $\hat{K}^{1/2}$ operator can be used to implement $K^{1/2}$ on the encoded qubit, by repeated application in the case that $l \neq 1$.

We still have to prove (8). Let the weight of a Pauli operator be the number of sites on which it acts nontrivially, and let us work modulo 8. Then (8) says that for any product $\pi = B_{c_1}^X \cdots B_{c_m}^X$, $\mathrm{wt}(\pi) \equiv 0$. This follows by induction on m. The case $m = 0$ is trivial. For the induction step, we first observe that if $\mathrm{wt}(\pi) \equiv 0$, then $\mathrm{wt}(\pi B_c^X) \equiv 0$ if and only if π and B_c^X share s sites with $s \equiv 0, 4$. But if $f_1, \ldots, f_j$ are those faces of c shared with some cell of π, then $s = \mathrm{wt}(B_{f_1}^Z \cdots B_{f_j}^Z)$. These faces are part of the 2D color lattice that forms the boundary of the cell c, from which it follows that $s \equiv 0, 4$ [19].

The Λ gate (1), known as the CNOT gate, is more straightforward. Imagine that we take two identical tetrahedral lattices and superpose them so that corresponding sites get very near. Then we apply $\hat{\Lambda}$, that is, we apply Λ pairwise. This can be achieved through single qubit operations and Ising interactions. As a result, it is easily checked that we get a Λ gate between the protected qubits.

As for measurements, the situation is the same as in any CSS code [20,21]. If we measure each physical qubit in the Z basis, then we are also performing a destructive measurement in the $\hat{Z}$ basis. Then nondestructive measurements of $\hat{Z}$ can be carried out performing a CNOT gate with the qubit to be measured as source and a $|\hat{0}\rangle$ state as target, and measuring the target destructively. Similarly, if we measure each physical qubit in the X basis we are performing a measurement in the $\hat{X}$ basis. We can admit faulty measurements, since the faulty measurement of a qubit is equivalent to an error prior to it. Thus the measuring process is as robust as the code itself and is topologically protected [7]. The results of the measurements must be classically processed to remove errors and recover the most probable code word.

Initialization is always a subtle issue in quantum computation, whether topological or not, and it certainly depends upon the physical implementation. In any case, even if perfectly pure $|\hat{0}\rangle$ or $|\hat{+}\rangle$ states cannot be provided, one can still purify them as much as necessary if their fidelity is above $\frac{1}{2}$. To do this, only the CNOT gate $\hat{\Lambda}$ and measurements in the $\hat{Z}$ and $\hat{X}$ bases are necessary.

As a concluding remark, we observe that the lattice that we have described so far unifies the strategies used in fault-tolerant computation, such as transversal operations, with the concept of a topologically protected quantum memory. Note that this approach is very different from the usual one in topological quantum computation, based on the braiding of non-Abelian anyons in a two-dimensional system. In fact, the topological order of the 3-dimensional system that we have described is Abelian.

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