Stochastic Dynamics of a Josephson Junction Threshold Detector

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We generalize the stochastic path integral formalism by considering Hamiltonian dynamics in the presence of general Markovian noise. Kramers' solution of the activation rate for escape over a barrier is generalized for non-Gaussian driving noise in both the overdamped and underdamped limit. We apply our general results to a Josephson junction detector measuring the electron counting statistics of a mesoscopic conductor. The activation rate dependence on the third current cumulant includes an additional term originating from the backaction of the measurement circuit.

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Detecting electron counting statistics has become a major experimental challenge in mesoscopic physics. First attempts to measure non-Gaussian effects in current noise have revealed that the detection problem is quite subtle. In particular, the experiment [1] found that the third current cumulant was not described by the simple theoretical prediction [2], but was masked by the influence of the measurement circuit causing an additional "cascade" correction [3,4]. Recent experiments demonstrated a measurement of the third current cumulant without cascade corrections [5], and the detection of individual electron counting statistics [6]. Stringent bandwidth requirements in measuring the third cumulant suggested that further experimental advances would require a new approach.

A conceptually different way to measure rare current fluctuations is with a threshold detector [7,8], the basic idea of which is analogous to a pole vault: A detection event occurs when the measured system variable exceeds a given value. A natural candidate for such a detector is a metastable system operating on an activation principle [9]. By measuring the rate of switching out of the metastable state, information about the statistical properties of the noise driving the system may be extracted. A threshold detector using an on-chip conductor which contains a region of negative differential resistance [8,10] was proposed by the authors and shown to be capable of measuring large deviations of current. Tobiska and Nazarov proposed a Josephson junction (JJ) threshold detector [7], the simplest variant of which (see Fig. 1) operates essentially in a Gaussian regime [11]. The third cumulant contribution is small [12] and may be extracted using the asymmetry of the switching rate with respect to bias current, as has been demonstrated in recent experiments [13,14].

In this Letter we solve Kramers' problem [9] of noise-activated escape from a metastable state beyond the Gaussian noise approximation and investigate how the measurement circuit affects threshold detection. Starting with general Hamiltonian-Langevin equations which include deterministic dynamics, dissipation, and fluctuations, we represent the solution as a stochastic path integral of Hamiltonian form [15,16] by doubling the number of degrees of freedom. In the weak damping case, the dynamics is dominated by energy diffusion, which we account for by a change of variables, enabling an effectively two-dimensional representation. We find the escape rate via an instanton calculation, and obtain a formal solution of Kramers' problem [9]. Applying these general results to a JJ threshold detector, we account for the influence of the measurement circuit and find that the cascade corrections are a consequence of the nonequilibrium character of the noise [17].

Hamiltonian-Langevin equations.—To reformulate and solve Kramers' problem beyond Gaussian noise, we introduce a theoretical framework that relies on a separation of time scales: Slow motion of a deterministic system on a time scale T_0 is affected by quickly fluctuating noise sources with correlation time $\tau_0 \ll T_0$. Quite generally, this situation can be described by Hamiltonian-Langevin (HL) equations. For instance, in a two-dimensional phase space (p, q) the equations of motion are

$$\dot{q} = \partial H(p, q)/\partial p + I_q, \qquad \dot{p} = -\partial H(p, q)/\partial q + I_p, \tag{1}$$

where $H(p, q)$ is the Hamiltonian that generates determi-

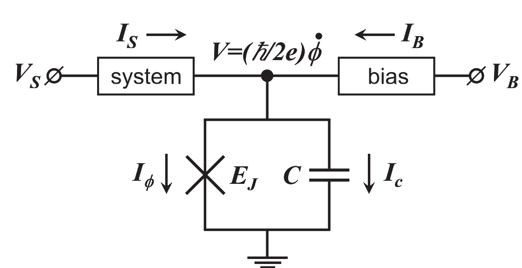

FIG. 1. Josephson junction (JJ) threshold detector: Simplified electrical circuit with the JJ (marked with an X) and mesoscopic system inserted. The driving noise of the system creates fluctuations of the center node voltage V, that can activate the JJ out of its supercurrent state, into its running dissipative state until it is recaptured.

nistic motion, and I_q and I_p are white noise sources. On the intermediate time scale t, such that $T_0 \gg t \gg \tau_0$, the sources are Markovian and fully characterized by the generating function $\mathcal{H}(\lambda_p, \lambda_q)$ of their cumulants (irreducible moments): $\langle\langle I_p^n I_q^m \rangle\rangle = \partial^n_{\lambda_p}\partial^m_{\lambda_q}\mathcal{H}(0,0)$.

While Langevin equations may be solved by standard methods in stochastic physics [18], the methods fail to account for cascade corrections [3,16] to higher-order cumulants, which originate from the effect of slow variations of p and q on the noise sources. Therefore, we follow the steps of Refs. [15,16] and represent the slow evolution of the system, governed by Eqs. (1), with the stochastic path integral (SPI) $P = \int \mathcal{D}\Lambda \int \mathcal{D}\mathbf{R}\exp(S)$, where the action S is given in the explicitly canonically invariant form as

$$S = \int dt'[-\Lambda \cdot \dot{\mathbf{R}} + \Lambda \cdot \{\mathbf{R}, H\} + \mathcal{H}(\Lambda, \mathbf{R})]. \quad (2)$$

Here $\mathbf{R} = (p, q)$ and $\Lambda = (\lambda_p, \lambda_q)$ are the sets of physical and canonically conjugated "counting" variables, respectively, and $\{\ldots\}$ denotes the Poisson bracket with respect to p and q. By fixing $\mathbf{R}$ in the final state of (2) we obtain the probability distribution $P(\mathbf{R})$, while fixing the final Λ variables turns the SPI into the moment generating function $P(\Lambda)$.

The large parameter $T_0/\tau_0 \gg 1$ allows the saddle-point evaluation of the SPI and thus requires solving Hamilton's equations of motion in the extended space [16]. There always exists a trivial solution $\Lambda = 0$ and $\dot{\mathbf{R}} = \{\mathbf{R}, H\} + \langle \mathbf{I}\rangle$, where $\mathbf{I} \equiv (I_p, I_q)$, that describes the "average" dynamics in physical space with a null action $S = 0$, giving the proper normalization of the distribution P. In the context of the generalized Kramers' problem, one has to find a nontrivial instanton solution $\Lambda_{\rm in}(\mathbf{R})$ of Eq. (2), for which the SPI gives a rate of the noise-activated escape from a localized state, $\Gamma \propto \exp(S_{\rm in})$ [19].

Energy diffusion and escape rate.—As an important case, let us consider quasiperiodic motion with the period $T_0 \gg \tau_0$. We change variables to energy $E = H(p, q)$ and "time" $s = s(p, q)$ [20], accompanied by a new set of counting variables (λ_E, λ_s), which are defined via $\lambda_p = \lambda_E\partial_p H + \lambda_s \partial_p s$ and $\lambda_q = \lambda_E \partial_q H + \lambda_s\partial_q s$. Using $\{E, H\} = 0$ and $\{s, H\} = 1$, we obtain $\Lambda \cdot \{\mathbf{R}, H\} = \lambda_s$. Fluctuations of the s variable increase the action without leading to escape; therefore, they can be neglected by choosing $\lambda_s = 0$. In the weak damping regime, one can set $s = t$, so that $\lambda_p = \dot{q}\lambda_E$ and $\lambda_q = -\dot{p}\lambda_E$. The action for the energy diffusion then reads [21]

$$S = \int dt'[-\lambda_E \dot{E} + \mathcal{H}(\lambda_E\dot{q}, -\lambda_E\dot{p})]. \quad (3)$$

To leading order in weak damping we can replace the generator $\mathcal{H}$ in Eq. (3) with its average over the oscillation period $\langle\mathcal{H}\rangle_E \equiv (1/T_0)\oint dt\mathcal{H}$, evaluated for fixed λ_E and E. Corrections in damping may be found by taking into account slow energy dissipation $\dot{E} = \partial_{\lambda_E}\mathcal{H}$ and $\dot{\lambda}_E = -\partial_E\mathcal{H}$, while averaging over the period T_0. We are interested in the instanton solution $\lambda_E = \lambda_{\rm in}(E)$ with $\langle\mathcal{H}\rangle_E = 0$ in the initial and final state [10]. Since the "Hamiltonian" $\langle\mathcal{H}\rangle_E$ is an integral of motion, we obtain an important result for the escape rate,

$$\log\Gamma = -\int \lambda_{\rm in} dE, \qquad \langle\mathcal{H}(\lambda_{\rm in}\dot{q}, -\lambda_{\rm in}\dot{p})\rangle_E = 0, \quad (4)$$

which formally solves Kramers' problem for arbitrary Markovian noise in the weak damping limit. Below we apply the theory outlined here to the stochastic dynamics of a Josephson threshold detector.

Josephson threshold detector.—The circuit in Fig. 1 shows the essential part of the detector comprised of the JJ with Josephson energy E_J, and the capacitor, C. The circuit is current biased with I_B through the macroscopic conductor and by the system current I_S, which is to be measured. According to Kirchhoff's law, the total bias current $I_S + I_B$ is equal to the sum of the Josephson current $I_\phi = (E_J/\Phi_0)\sin\phi$ where $\Phi_0 = \hbar/2e$, and the displacement current $I_C = C\dot{V}$. This leads to the equation of motion for the superconducting phase ϕ,

$$C\Phi_0^2\ddot{\phi} + E_J\sin\phi = \Phi_0(I_S + I_B), \quad (5)$$

where we used the relation $V = \Phi_0\dot{\phi}$.

To simplify the following analysis and concentrate on our main message, we make a number of assumptions, most of which will be relaxed later. First, we consider an Ohmic system and bias resistor, so that $\langle I_S\rangle = J_S - G_S V$ and $\langle I_B\rangle = J_B - G_B V$, where G_S, G_B are the system and bias conductances, and the constant currents $J_S = G_S V_S$, $J_B = G_B V_B$ are just tunable parameters. The bias resistor, being a macroscopic system, creates Gaussian Nyquist noise $\langle\langle I_B^2\rangle\rangle = 2TG_B$. We further assume a high-impedance circuit, so that the back flow part of the bias current $G_B V$ and the Nyquist noise may be neglected, $I_B = J_B$, which we refer to as the ideal detection scheme. Then Eq. (5) can be rewritten as a set of HL equations (1) for the phase variable ϕ and canonically conjugated momentum $p = \Phi_0 CV$,

$$\dot{\phi} = p/m, \qquad \dot{p} = -\partial U/\partial\phi + \Phi_0(I_S - J_S), \quad (6)$$

with "mass" $m = \Phi_0^2 C$.

Equations (6) describe the motion of a "particle" in the tilted periodic potential

$$U(\phi) = -E_J\cos\phi - \Phi_0(J_S + J_B)\phi, \quad (7)$$

stimulated by the dissipative part $I_S - J_S$ of the system's current. For later convenience, we define the normalized total bias current as $\mathcal{J} = \Phi_0(J_S + J_B)/E_J$. In what follows, we investigate noise-activated escape from the metastable supercurrent state, where $\langle V\rangle = 0$.

Weak damping regime.—Here the system conductance is small, $G_S \ll \omega_{\rm pl}C$, so the phase oscillates with the plasma frequency $\omega_{\rm pl} = \Omega_J(1 - \mathcal{J})^{1/4}$, where $\Omega_J = (E_J/\Phi_0^2 C)^{1/2}$. The energy relaxes slowly with the rate G_S/C to the local potential minimum. We further assume the separation of time scales, $1/\tau_0 \sim \max\{eV_S, T\} \gg \hbar\omega_{\rm pl}$, so that the noise source I_S is Markovian. Comparing (6) and (1), we identify $q = \phi$, $I_p = \Phi_0(I_S - J_S)$ and $I_q = 0$, so the equations for the escape rate and "instanton line" read

$$\log\Gamma = -\Phi_0^{-1}\int \lambda_{\rm in} dE, \qquad \langle \mathcal{H}(\lambda_{\rm in}\dot\phi)\rangle_E = 0, \tag{8}$$

where $\mathcal{H}$ is the generator of the cumulants of the dissipative part of the system current, $I_S - J_S$, and Φ_0 plays the role of an effective charge.

We first consider the system in thermal equilibrium as a checkpoint of the theory. The cumulant generator acquires the simple form $\mathcal{H} = -\Phi_0 G_S\dot\phi(\lambda_{\rm in}\dot\phi) + TG_S(\lambda_{\rm in}\dot\phi)^2$, where the first term comes from the linear response of the system current $I_S - J_S = -G_S V$ to the potential $V = \Phi_0\dot\phi$, while the second term is the Gaussian Nyquist noise contribution, $\langle\langle I_S^2\rangle\rangle = 2TG_S$. Averaging $\mathcal{H}$ over the period of oscillations, we observe that the term $\langle\dot\phi^2\rangle_E$ cancels, so that $\lambda_{\rm in} = \Phi_0/T$. Using Eq. (8) we obtain Kramers' well-known formula for the rate of thermally activated escape

$$\log\Gamma = -\Delta U/T, \tag{9}$$

where the potential threshold ΔU is a function of normalized bias $\mathcal{J}$. For the potential (7) one obtains $\Delta U/E_J = 2(1-\mathcal{J}^2)^{1/2} - 2\mathcal{J}\arccos\mathcal{J}$, see Fig. 2. Limiting values are $\Delta U/E_J = 2$ for $\mathcal{J} = 0$, while as $\mathcal{J}\to 1$, $\Delta U/E_J \approx (4\sqrt{2}/3)(1-\mathcal{J})^{3/2}$.

The above example shows that the argument of $\mathcal{H}$ is small. Indeed, we estimate $\dot\phi \sim \omega_{\rm pl}$, so $e\dot\phi\lambda_{\rm in} \sim \hbar\omega_{\rm pl}/T \ll 1$ due to the separation of time scales. Therefore, away from equilibrium we can expand $\langle\mathcal{H}\rangle_E$

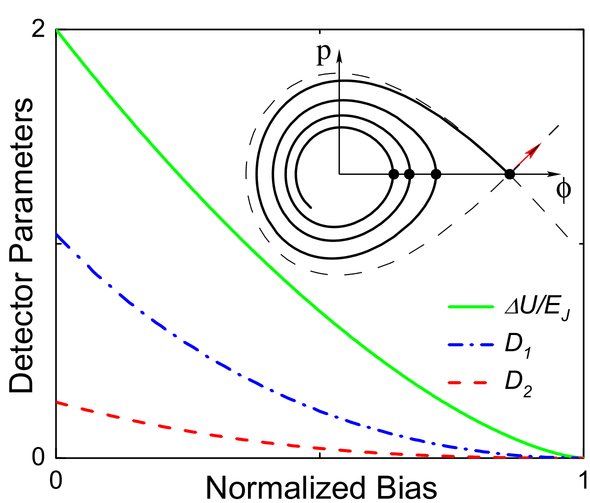

FIG. 2 (color online). The normalized potential threshold $\Delta U/E_J$ and dimensionless factors D_1 and D_2, are plotted versus the normalized current bias $\mathcal{J} = \Phi_0(J_S + J_B)/E_J$. The inset shows the phase space plot of a trajectory for weak damping (thick line) which leads to the escape, and the separatrix (dashed line). The turning points ϕ_0 are shown by the dots.

in Eq. (8), average the result over the period of oscillations, and invert the series for $\lambda_{\rm in}$. To second order in $\lambda_{\rm in}$ we again obtain Kramers' formula [9] with the temperature T replaced with the effective temperature

$$T_{\rm eff} \equiv \langle\langle I_S^2\rangle\rangle/2G_S \tag{10}$$

of the nonequilibrium noise.

To find the third cumulant correction to the Kramers' formula, we expand $\langle\mathcal{H}(\dot\phi\lambda_{\rm in})\rangle_E$ in $\lambda_{\rm in}$ and the voltage V, and collect all terms to next order in small parameter $\hbar\omega_{\rm pl}/T_{\rm eff}$. Two terms which contain $\langle\langle I_S^3\rangle\rangle$ and $\partial_{V_S}\langle\langle I_S^2\rangle\rangle$ come with the factor $\langle\dot\phi^3\rangle_E$ [22]. This factor vanishes to leading order in damping, because for fixed energy $\dot\phi$ is an odd function of time over the oscillation period. Therefore, we evaluate the average $\langle\dot\phi^3\rangle_E$ by taking into account slow variation of the energy. This is accomplished by taking functional variations of the action to find $\lambda_{\rm in} = \Phi_0/T_{\rm eff}$ and $\dot E = \Phi_0^2 G_S\dot\phi^2$. We skip a number of straightforward steps and present the result for the nonequilibrium escape rate

$$\log\Gamma = -\frac{\Delta U}{T_{\rm eff}} + \frac{D_1\Phi_0 E_J\langle\langle I_S^3\rangle\rangle_{\rm tot}}{CT_{\rm eff}^3}, \tag{11a}$$

$$D_1 = \int\frac{dE}{E_J}\frac{\langle(\phi_0-\phi)\dot\phi^2\rangle_E}{3\langle\dot\phi^2\rangle_E}, \tag{11b}$$

where the dimensionless factor $D_1(\mathcal{J})$ is a characteristic of the detector. Appearing in the integral (11b) is one of the turning points of the oscillating phase, ϕ_0, defined by the solution of $U(\phi_0) = E$ that is nearest the top of the potential. For the potential (7) $D_1(0) = \pi/3$ and for strong bias $\mathcal{J}\to 1$ we obtain $D_1(\mathcal{J}) \approx 0.8(1-\mathcal{J})^2$. The parameter D_1, evaluated numerically, is shown in Fig. 2.

The total third current cumulant

$$\langle\langle I_S^3\rangle\rangle_{\rm tot} = \langle\langle I_S^3\rangle\rangle - 3T_{\rm eff}\partial_{V_S}\langle\langle I_S^2\rangle\rangle \tag{12}$$

is taken at $V = 0$ and contains a correction originating from the slowly varying second current cumulant. This contribution is analogous to the cascade correction [3,4] directly observed in third cumulant [1]. The correction is generally not small and may even change the sign of the total cumulant. For instance, for systems far from equilibrium $\langle\langle I_S^n\rangle\rangle = F_n\langle I_S\rangle$, where F_n are the dimensionless Fano factors, we obtain $\langle\langle I_S^3\rangle\rangle_{\rm tot} = (F_3 - 3F_2^2/2)\langle I_S\rangle$.

Overdamped regime.—In this regime the conductance is large, $G_S \gg \omega_{\rm pl}C$, and the dynamics is entirely due to slow phase relaxation with the rate $\omega_{\rm pl}^2C/G_S$. Therefore, we can set $p = m\dot\phi$ and $\lambda_q = 0$, and neglect the first term in the action (2), so that the action reads $S = \int dt[\mathcal{H}(\Phi_0\lambda_p, \dot\phi) - \lambda_p\partial_\phi U]$, where again $\mathcal{H}$ is the generator of the cumulants of the dissipative part of the system current, $I_S - J_S$.

The following analysis is analogous to that of the weak damping regime. We first expand $\mathcal{H}$ to second order in λ_p and find from the equations of motion that $\dot\phi = T_{\rm eff}\lambda_p$ and

$\lambda_p = \partial_\phi U/(\Phi_0^2 T_{\rm eff} G_S)$. Substituting these results back to the action we immediately obtain Kramers' formula [9] with T replaced with $T_{\rm eff}$. Next we observe that the argument of $\mathcal{H}$ is small, namely $e\Phi_0\lambda_p \sim \hbar\omega_{\rm pl}^2 C/G_S T_{\rm eff} \ll 1$ due to the separation of time scales requirement. We collect all the terms to third order in this parameter and evaluate them perturbatively using the above results for λ_p and $\dot\phi$. We finally obtain

$$\log\Gamma = -\frac{\Delta U}{T_{\rm eff}} + \frac{D_2 E_J^2 \langle\langle I_S^3\rangle\rangle_{\rm tot}}{\Phi_0 T_{\rm eff}^3 G_S^2}, \tag{13a}$$

$$D_2 = (1/6)\int d\phi(\partial_\phi U/E_J)^2, \tag{13b}$$

where the total cumulant $\langle\langle I_S^3\rangle\rangle_{\rm tot}$ is given by Eq. (12), and $D_2(\mathcal{J})$ is a dimensionless detector property. For the potential (7) it is given by $D_2 = (1/6)(1 + 2\mathcal{J}^2)\times \arccos\mathcal{J} - (1/2)\mathcal{J}(1-\mathcal{J}^2)^{1/2}$, see Fig. 2. Limits are $D_2(0) = \pi/12$, and as $\mathcal{J}\to 1$, $D_2(\mathcal{J}) \approx 0.25(1-\mathcal{J})^{5/2}$.

Discussion.—We now remark on the application of our results to the detection of non-Gaussian fluctuations. It is evident from Fig. 2 that $D_{1,2} \ll \Delta U/E_J$ as $\mathcal{J}\to 1$. Therefore, in the strong bias regime the third cumulant contribution in (11) and (13) is suppressed compared to the Kramers' term $S_0 = \Delta U/T_{\rm eff}$. But even for a relatively weak bias, when $D_{1,2}\sim 1$, non-Gaussian effects are small. Indeed, we estimate the correction as $(\mathcal{R}_{\rm wd}/\mathcal{Q})S_0$, where the JJ quality factor $\mathcal{Q} = C\omega_{\rm pl}/G_S > 1$ in the weak damping regime (wd), and the ratio $\mathcal{R}_{\rm wd} = \hbar\omega_{\rm pl}/T_{\rm eff} < 1$ due to the separation of time scales. Similarly, in the strong damping regime (sd), $\mathcal{Q} < 1$, the correction is of order $\mathcal{R}_{\rm sd}S_0$, where $\mathcal{R}_{\rm sd} = \hbar\omega_{\rm pl}^2 C/G_S T_{\rm eff} < 1$ due to the separation of time scales. Since S_0 itself cannot be too large for the escape to be detected, in experiments one should try to saturate the above inequalities and use the asymmetry of the third cumulant (12) as a function of the current bias J_B.

Next, we briefly discuss nonideal detection and nonlinear effects. The finite conductance of the bias resistor G_B contributes to the total conductance of the circuit $G_{\rm tot} = G_S + G_B$ and the Nyquist noise $\langle\langle I_B^2\rangle\rangle = 2TG_B$ adds to the total noise power. Therefore, in Eqs. (11)–(13) one has to replace G_S with $G_{\rm tot}$ and the effective temperature with $\tilde T = (G_S T_{\rm eff} + G_B T)/G_{\rm tot}$. The nonlinearity of the system current leads to the correction $(\lambda_p/2)\times(\Phi_0\dot\phi)^2\partial_{V_S}^2\langle I_S\rangle$ to $\mathcal{H}$, and thereby to an additional contribution to the total third cumulant, $\langle\langle I_S^3\rangle\rangle_{\rm tot} = \langle\langle I_S^3\rangle\rangle - 3\tilde T\partial_{V_S}\langle\langle I_S^2\rangle\rangle + 3\tilde T^2\partial_{V_S}^2\langle I_S\rangle$, in both transport regimes considered above. Quite remarkably, this total third cumulant is related via $\langle\langle I_S^3\rangle\rangle_{\rm tot} = 3C^2 G_{\rm tot}\langle\langle V^3\rangle\rangle$ to the instantaneous fluctuations of the voltage V [23].

We finally note that in the experiment of Ref. [13] the circuit corrections have not been observed, which we explain by a very low impedance of the circuit, $G_S/G_{\rm tot} \ll 1$, operating in the weak damping regime with $\mathcal{Q} = C\omega_{\rm pl}/G_{\rm tot} = 2.5$. Indeed, in this case $\tilde T = (G_S/2G_{\rm tot})eV_S$ (assuming $T = 0$); therefore, the circuit corrections are suppressed by a small factor $G_S/G_{\rm tot}$. On the other hand, the first term in Eq. (11a) is $S_0 = \Delta U/\tilde T$ and the second term can be estimated as $(G_{\rm tot}/G_S)(\mathcal{R}_{\rm wd}/\mathcal{Q})S_0$, where $\mathcal{R}_{\rm wd} = \hbar\omega_{\rm pl}/eV_S$ in this case. Therefore the relative contribution of the third cumulant to the total action increases by the factor $G_{\rm tot}/G_S$, which makes it favorable to use a low-impedance circuit.

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