# Test of the Anti–de Sitter-Space/Conformal-Field-Theory Correspondence Using High-Spin Operators

M. K. Benna,[1] S. Benvenuti,[1] I. R. Klebanov,[1,2] and A. Scardicchio[1,2]

[1]*Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA*

[2]*Princeton Center for Theoretical Physics, Princeton University, Princeton, New Jersey 08544, USA*

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In two remarkable recent papers the planar perturbative expansion was proposed for the universal function of the coupling appearing in the dimensions of high-spin operators of the $\mathcal{N} = 4$ super Yang-Mills theory. We study numerically the integral equation derived by Beisert, Eden, and Staudacher, which resums the perturbative series. In a confirmation of the anti–de Sitter-space/conformal-field-theory (AdS/CFT) correspondence, we find a smooth function whose two leading terms at strong coupling match the results obtained for the semiclassical folded string spinning in $AdS_5$. We also make a numerical prediction for the third term in the strong coupling series.

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*Introduction.*—The dimensions of high-spin operators are important observables in gauge theories. It is well known that the anomalous dimension of a twist-2 operator grows logarithmically for large spin S,

$$\Delta - \mathrm{S} = f(g)\ln \mathrm{S} + O(\mathrm{S}^0), \qquad g = \frac{\sqrt{g_{\mathrm{YM}}^2 N}}{4\pi}. \qquad (1)$$

This effect is important for the physics of QCD; it determines the behavior of parton distribution functions as the Bjorken-$x$ parameter approaches 1 [1]. The logarithmic growth of $\Delta - \mathrm{S}$ was demonstrated early on at 1-loop order [1] and at 2 loops [2] where a cancellation of $\ln^3\mathrm{S}$ terms occurs. There are solid arguments that (1) holds to all orders in perturbation theory [3,4], and that it also applies to high-spin operators of twist greater than two [5]. The universal function of coupling $f(g)$ also measures the anomalous dimension of a cusp in a lightlike Wilson loop, and is of definite physical interest in QCD.

There has been significant interest in determining $f(g)$ in the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. This is partly due to the fact that the anti–de Sitter-space/conformal-field-theory (AdS/CFT) correspondence [6] relates the dimensions of operators in this gauge theory to energies of corresponding objects in type IIB string theory on $AdS_5 \times S^5$. The object dual to a high-spin twist-2 operator is a folded straight string spinning around the center of $AdS_5$ space [7]. For large $g$ the dual $AdS_5 \times S^5$ background becomes weakly curved, and semiclassical calculations of the spinning string energy become reliable. This gives the prediction that $f(g) \to 4g$ at strong coupling [7]. The same result was obtained from studying the cusp anomaly using string theory methods [8]. Furthermore, the semiclassical expansion for the spinning string energy predicts the following correction [9]:

$$f(g) = 4g - \frac{3\ln 2}{\pi} + O(1/g). \qquad (2)$$

It is of obvious interest to confirm these explicit predictions of string theory using extrapolation of the perturbative expansion for $f(g)$ provided by the gauge theory.

Explicit perturbative calculations are quite formidable, and until recently were available only up to 3-loop order [10,11]:

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 + O(g^8). \qquad (3)$$

Kotikov, Lipatov, Onishchenko, and Velizhanin (KLOV) [10] extracted the $\mathcal{N} = 4$ answer from the QCD calculation of [12] using their proposed transcendentality principle stating that each expansion coefficient has terms of the same degree of transcendentality.

Recently, the methods of integrability in AdS/CFT (For earlier work on integrability in gauge theories, see [13–15].) [16], prompted in part by [7,17], have led to dramatic progress in studying the weak coupling expansion. In the beautiful paper by Beisert, Eden, and Staudacher [21], which followed closely the important earlier work in [18,19], an integral equation that determines $f(g)$ was proposed, yielding an expansion of $f(g)$ to an arbitrary desired order. The expansion coefficients obey the KLOV transcendentality principle. In an independent remarkable paper by Bern, Czakon, Dixon, Kosower, and Smirnov [20], an explicit calculation led to a value of the 4-loop term,

$$-16\left(\frac{73}{630}\pi^6 + 4\zeta(3)^2\right)g^8, \qquad (4)$$

which agrees with the idea advanced in [20,21] that the exact expansion of $f(g)$ is related to that found in [18] simply by multiplying each $\zeta$-function of an odd argument by an $i$, $\zeta(2n+1) \to i\zeta(2n+1)$. The integral equation of [21] generates precisely this perturbative expansion for $f(g)$.

 

A crucial property of the integral equation proposed in [21] is that it is related through integrability to the "dressing phase" in the magnon S-matrix, whose general form was deduced in [22,23]. In [21] a perturbative expansion of the phase was given, which starts at the 4-loop order, and at strong coupling coincides with the earlier results from string theory [19,22,24–26]. An important requirement of crossing symmetry [27] is satisfied by this phase, and it also satisfies the KLOV transcendentality priciple. Therefore, this phase is very likely to describe the exact magnon S-matrix at any coupling [21], which constitutes remarkable progress in the understanding of the $\mathcal{N} = 4$ SYM theory, and of the AdS/CFT correspondence.

The papers [20,21] thoroughly studied the perturbative expansion of $f(g)$ which follows from the integral equation. Although the expansion has a finite radius of convergence, as is customary in certain planar theories (see, for example, [28]), it is expected to determine the function completely. Solving the integral equation of [21] is an efficient tool for attacking this problem. In this Letter we solve the integral equation numerically at intermediate coupling, and show that $f(g)$ is a smooth function that approaches the asymptotic form (2) predicted by string theory for $g > 1$. The two leading strong coupling terms match those in (2) with high accuracy. This constitutes a remarkable confirmation of the AdS/CFT correspondence for this nonsupersymmetric observable.

*Numerical study of the integral equation.*—The cusp anomalous dimension $f(g)$ can be written as [21,18,29]

$$f(g) = 16g^2 \hat{\sigma}(0), \tag{5}$$

where $\hat{\sigma}(t)$ obeys a certain integral equation. In terms of the function $s(t) = \frac{e^t - 1}{t} \hat{\sigma}(t)$ the integral equation is

$$s(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \frac{t'}{e^{t'} - 1} s(t'), \tag{6}$$

with the kernel given by [21]

$$K(t, t') = K^{(m)}(t, t') + 2K^{(c)}(t, t'). \tag{7}$$

The main scattering kernel $K^{(m)}$ of [18] is

$$K^{(m)}(t, t') = \frac{J_1(t)J_0(t') - J_0(t)J_1(t')}{t - t'}, \tag{8}$$

and the dressing kernel $K^{(c)}$ is defined as the convolution

$$K^{(c)}(t, t') = 4g^2 \int_0^\infty dt'' K_1(t, 2gt'') \frac{t''}{e^{t''} - 1} K_0(2gt'', t'), \tag{9}$$

where $K_0$ and $K_1$ denote the parts of the kernel that are even and odd, respectively, under change of sign of $t$ and $t'$:

$$K_0(t, t') = \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'} \sum_{n=1}^{\infty} (2n - 1)J_{2n-1}(t)J_{2n-1}(t'), \tag{10}$$

$$K_1(t, t') = \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} = \frac{2}{tt'} \sum_{n=1}^{\infty} (2n)J_{2n}(t)J_{2n}(t'). \tag{11}$$

Both $K^{(m)}$ and $K^{(c)}$ can conveniently be expanded as sums of products of functions of $t$ and functions of $t'$:

$$K^{(m)}(t, t') = K_0(t, t') + K_1(t, t') = \frac{2}{tt'} \sum_{n=1}^{\infty} nJ_n(t)J_n(t'), \tag{12}$$

and

$$K^{(c)}(t, t') = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{8n(2m - 1)}{tt'} Z_{2n,2m-1} J_{2n}(t)J_{2m-1}(t'). \tag{13}$$

This suggests writing the solution in terms of linearly independent functions as

$$s(t) = \sum_{n \ge 1} s_n \frac{J_n(2gt)}{2gt}, \tag{14}$$

so that the integral equation becomes a matrix equation for the coefficients $s_n$. The desired function $f(g)$ is now $f(g) = 8g^2 s_1$.

It is convenient to define the matrix $Z_{mn}$ as

$$Z_{mn} \equiv \int_0^\infty dt \frac{J_m(2gt)J_n(2gt)}{t(e^t - 1)}. \tag{15}$$

Using the representations (12) and (13) of the kernels and (14) for $s(t)$, the integral equation above is now of the schematic form

$$s_n = h_n - \sum_{m \ge 1} (K^{(m)}_{nm} + 2K^{(c)}_{nm}) s_m, \tag{16}$$

whose solution is

$$s = \frac{1}{1 + K^{(m)} + 2K^{(c)}} h. \tag{17}$$

The matrices are

$$K^{(m)}_{nm} = 2(NZ)_{nm}, \tag{18}$$

$$K^{(c)}_{nm} = 2(CZ)_{nm}, \tag{19}$$

$$C_{nm} = 2(PNZQN)_{nm}, \tag{20}$$

where $Q = \mathrm{diag}(1, 0, 1, 0, \ldots)$, $P = \mathrm{diag}(0, 1, 0, 1, \ldots)$,

$N = \mathrm{diag}(1, 2, 3, \ldots)$, and the vector $h$ can be written as $h = (1 + 2C)e^{\mathrm{T}}$, where $e = (1, 0, 0, \ldots)$. The crucial point for the numerics to work is that the matrix elements of $Z$ decay sufficiently fast with increasing $m, n$ (they decay like $e^{-\max(m,n)/g}$). For intermediate $g$ (say $g < 20$) we can work with moderate size $d$ by $d$ matrices, where $d$ does not have to be much larger than $g$. The integrals in $Z_{nm}$ can be obtained numerically without much effort and so we can solve for the $s_n$. We find that the results are stable with respect to increasing $d$.

Even though at strong coupling all elements of $Z_{nm}$ are of the same order in $1/g$, those far from the upper left corner are numerically small. This last fact makes the numerics surprisingly convergent even at large $g$ and, moreover, gives some hope that the analytic form of the strong coupling expansion of $f(g)$ could be obtained from a perturbation theory for the matrix equation.

Therefore, when formulated in terms of the $Z_{mn}$, the problem becomes amenable to numerical study at all values of the coupling. We find that the numerical procedure converges rather rapidly, and truncates the series expansions of $s(t)$ and of the kernel after the first 30 orders of Bessel functions.

The function $f(g)$ is the lowest curve plotted in Fig. 1. For comparison, we also plot $f_m(g)$ which solves the integral equation with kernel $K^{(m)}$ [18], and $f_0(g)$ which solves the integral equation with kernel $K^{(m)} + K^{(c)}$. Clearly, these functions differ at strong coupling. The function $f(g)$ is monotonic and reaches the asymptotic, linear form quite early, for $g \simeq 1$. We can then study the asymptotic, large $g$ form easily and compare it with the prediction from string theory. The best fit result (using the range $2 < g < 20$) is

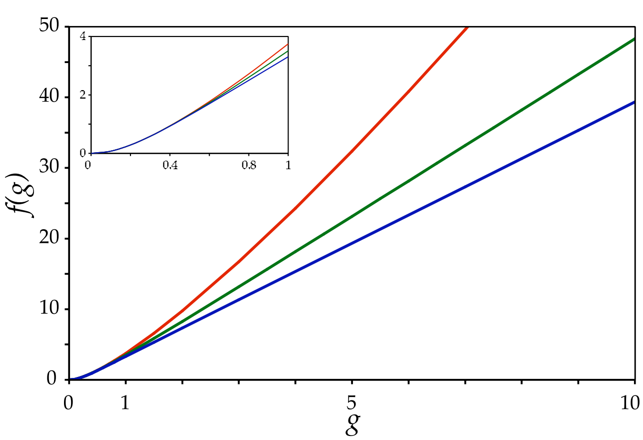


FIG. 1 (color online). Plot of the solutions of the integral equations: $f_m(g)$ for the Eden and Staudacher kernel [18] $K^{(m)}$ (upper curve, red), $f_0(g)$ for the kernel $K^{(m)} + K^{(c)}$ (middle curve, green), and $f(g)$ for the BES kernel $K^{(m)} + 2K^{(c)}$ (lower curve, blue). Notice the different asymptotic behaviors. The inset shows the three functions in the crossover region $0 < g < 1$.

$$f(g) = (4.000\,000 \pm 0.000\,001)g - (0.661\,907 \pm 0.000\,002) - \frac{0.0232 \pm 0.0001}{g} + \ldots. \quad (21)$$

The first two terms are in remarkable agreement with the string theory result (2), while the third term is a numerical prediction for the $1/g$ term in the strong coupling expansion. The coefficients in (21) are obtained by fitting our results to a polynomial in $1/g$ with 5 parameters. The error in the second (third) term is estimated by fitting the numerical data after the first (respectively, first and second) coefficients have been fixed to their string theoretic values (2). If one does not fix any coefficient the error in the third term is somewhat larger (4% rather than 0.5%) while the error in the second is still negligible. The value $0.0232 \pm 0.0001$ for the $1/g$ term is obtained by fitting the data after fixing the first two terms to their string theoretic values (2). The 3-parameter fit gives the same central value but with a bigger error (4% instead of 0.5%)], which may perhaps be checked one day against a two-loop string theory calculation. It is worth mentioning that we obtain a very good fit to the numerical results without introducing any anomalous terms like $\log g/g$.

We do not need to restrict the numerical analysis to real values of $g$; complex values of $g$ are of interest as well. In [21] it was argued that the dressing phase has singularities at $g \approx \pm in/4$, for $n = 1, 2, 3, \ldots$. Also, their analysis of the small $g$ series shows that there are square-root branch points in $f(g)$ at $g = \pm i/4$. Perhaps, this is related to the cuts in the giant-magnon dispersion relations [26,30–33], for momenta close to $\pi$. Our numerical results indeed indicate branch points at $g \approx \pm i/4$, $\pm i/2$ with exponent $1/2$. Beyond that we observe oscillations of both the real and imaginary parts of $f(g)$ for nearly imaginary $g$. Further work is needed to elucidate the analytical structure of $f(g)$.

*Discussion.*—A very satisfying result of this Letter is that the Beisert, Eden, and Staudacher (BES) integral equation yields a smooth universal function $f(g)$ whose strong coupling expansion is in excellent numerical agreement with the spinning string predictions of [7,9]. This provides a highly nontrivial confirmation of the AdS/CFT correspondence.

The agreement of this strong coupling expansion was anticipated in [21] based on a similar agreement of the dressing phase. However, some concerns about this argument were raised in [20] based on the slow convergence of the numerical extrapolations. Luckily, our numerical methods employed in solving the integral equation converge rapidly and produce a smooth function that approaches the asymptotics (2). The crossover region of $f(g)$ where it changes from the perturbative to the linear behavior lies right around the radius of convergence, $g_c = 1/4$, corresponding to $g_{\mathrm{YM}}^2 N = \pi^2$. For $N = 3$, this would correspond to $\alpha_s \sim 0.25$.

The qualitative structure of the interpolating function $f(g)$ is quite similar to that involved in the circular Wilson loop, where the conjectured exact result [34,35] is $\ln(\frac{I_1(4\pi g)}{2\pi g})$. The above function is analytic on the complex plane, with a series of branch cuts along the imaginary axis, and an essential singularity at infinity. The function $f(g)$ is also expected to have an infinite number of branch cuts along the imaginary axis, and an essential singularity at infinity [21]. We found numerically the presence, in $f(g)$, of the first two branch cuts on the imaginary axis, starting at $g = \pm\frac{ni}{4}$, $n = 1, 2$. The first of them, which also occurs for the giant magnon with maximal momentum $p = \pi$, agrees with the summation of the perturbative series [21].

It is remarkable that the integral equation of [21] allows $f(g)$, which is not an observable protected by supersymmetry, to be solved for. Hopefully, this paves the way to finding other observables as functions of the coupling in the planar $\mathcal{N} = 4$ SYM theory.

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