Are $f(R)$ Dark Energy Models Cosmologically Viable?

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All $f(R)$ modified gravity theories are conformally identical to models of quintessence in which matter is coupled to dark energy with a strong coupling. This coupling induces a cosmological evolution radically different from standard cosmology. We find that, in all $f(R)$ theories where a power of R is dominant at large or small R (which include most of those proposed so far in the literature), the scale factor during the matter phase grows as $t^{1/2}$ instead of the standard law $t^{2/3}$. This behavior is grossly inconsistent with cosmological observations (e.g., Wilkinson Microwave Anisotropy Probe), thereby ruling out these models even if they pass the supernovae test and can escape the local gravity constraints.

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The late-time accelerated cosmic expansion is a major challenge to cosmology [1]. It can be due to an exotic component with sufficiently negative pressure, dark energy (DE), or, alternatively, to a modification of gravity, no longer described by general relativity. Examples of such modified gravity DE models are theories where the Ricci scalar R in the Lagrangian is replaced by some function $f(R)$, e.g., inverse powers R^{-n} [2,3]. Although these models exhibit a natural acceleration mechanism, criticisms emphasized their inability to pass solar system constraints [4]. Indeed, $f(R)$ theories correspond to scalar-tensor gravity with a vanishing Brans-Dicke parameter $\omega_{\rm BD}$ [5]. However, one could, in principle, build models with a very short interaction range (e.g., adding a R^2 term [6,7]) or assume decoupling of the baryons from modified gravity. Since these models could pass local gravity constraints, it is important to assess their cosmological viability: This is the aim of this Letter. We will consider models of the form $f(R) = R - \mu^{2(n+1)}/R^n$ for all μ, n in which $df/dR > 0$ for all R. In these models, the scale factor $a(t)$ expands as $t^{1/2}$ instead of the conventional $t^{2/3}$ behavior during the matter phase that *precedes* the final accelerated stage (in contrast with *inflationary* models such as Starobinsky's R^2 one [8]). This would lead to inconsistencies with the observed distance to the cosmic microwave background (CMB), the large scale structure (LSS) formation, and the age of the Universe. This crucial fact appears to have been overlooked so far.

Consider the general action in the Jordan frame (JF)

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}f(R) + \mathcal{L}_m\right], \tag{1}$$

where $\kappa^2 \equiv 8\pi G$ (G is the gravitational constant). For a flat Friedmann-Robertson-Walker metric, the equations are given by

$$\begin{aligned} 3FH^2 &= (RF - f)/2 - 3H\dot{F} + \kappa^2\rho_m, \\ 2F\dot{H} &= -\ddot{F} + H\dot{F} - \kappa^2(\rho_m + p_m), \end{aligned} \tag{2}$$

where $F \equiv \partial f/\partial R$, $H \equiv \dot{a}/a$, and ρ_m and p_m represent the energy density and the pressure of a perfect fluid, respectively, obeying the standard conservation equation. These equations coincide with a scalar-tensor Brans-Dicke theory with a potential and vanishing $\omega_{\rm BD}$ [9,10].

Under the conformal transformation $\tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}$, $2\omega = \log F$, one obtains the Einstein frame (EF) action:

$$S_E = \int d^4x\sqrt{-\tilde{g}}\left[\frac{R(\tilde{g})}{2\kappa^2} - \frac{1}{2}(\tilde{\nabla}\phi)^2 - V(\phi) + \tilde{\mathcal{L}}_m(\phi)\right], \tag{3}$$

where $\phi \equiv \sqrt{6}\omega/\kappa$ and $V = \mathrm{sgn}(F)(RF - f)/2\kappa^2F^2$ (all tilded quantities are in the EF). The conformal transformation is singular for $F = 0$ and trivial for $F = \mathrm{const}$, so we will consider only positive-definite nonconstant forms of F. Quantities in the two frames are related as follows:

$$\tilde{\rho}_m = \rho_m e^{-4\omega}, \quad \tilde{p}_m = p_m e^{-4\omega}, \quad d\tilde{t} = e^{\omega}dt, \quad \tilde{a} = e^{\omega}a. \tag{4}$$

Although we will work mainly in the EF, we checked all numerical and analytical results directly in the JF as well. In the EF, the field ϕ and the fluid satisfy the standard gravitational and conservation equations:

$$\ddot{\phi} + 3\tilde{H}\dot{\phi} + V_{,\phi} = \sqrt{2/3}\kappa\beta(\tilde{\rho}_m - 3\tilde{p}_m), \tag{5}$$

$$\dot{\tilde{\rho}}_m + 3\tilde{H}(\tilde{\rho}_m + \tilde{p}_m) = -\sqrt{2/3}\kappa\beta\dot{\phi}(\tilde{\rho}_m - 3\tilde{p}_m), \tag{6}$$

where the coupling β is given by

$$\beta = 1/2, \tag{7}$$

regardless of the form of $f(R)$. Then the strength of the coupling between the field and the fluid is uniquely determined in all $f(R)$ gravity theories. A dimensionless strength of order unity means that matter feels an additional scalar force as strong as gravity itself. Note that β is related to $\omega_{\rm BD}$ via the relation $\beta = [3/4(2\omega_{\rm BD} + 3)]^{1/2}$. The dynamics of the system depends upon the form of the potential $V(\phi)$, i.e., the choice of $f(R)$. For theories in which $f(R) = -\mu^{2(n+1)}R^{-n}$ ($n \neq -1$, 0, and negative n are also included), the potential in EF is a pure exponential

$$V(\phi) = A\exp(-\lambda\kappa\phi), \tag{8}$$

where $\lambda = (\sqrt{6}/3)\frac{n+2}{n+1}$ and $A = \mu^2(n+1)/2\kappa^2 n|n|^{1/(n+1)}$. The condition $F > 0$ implies $A > 0$ except for $-1 < n < 0$: In this case, since the potential becomes negative, we analyze directly the JF. In the EF, the R^{-n} model corresponds to a coupled DE scenario studied in Refs. [11,12] with the coupling (7). We first discuss the main properties of this exponential potential and then extend them to the general case.

As shown in Ref. [12] for all values of n outside $(-1, 0)$, the system has one and only one global attractor solution, a scalar-field-dominated solution with an energy fraction $\tilde{\Omega}_\phi = 1$. This solution appears when the potential term in Eq. (5) dominates over the coupling term on the right-hand side and is, therefore, independent of the coupling. On this attractor, the scale factor evolves as $\tilde{a} \sim \tilde{t}^{2/[3(1+\tilde{w}_{\rm eff})]}$, where the effective equation of state (EOS) is $\tilde{w}_{\rm eff} = -1 + \lambda^2/3 = -1 + [2(2+n)^2/9(1+n)^2]$. This can be identified with the acceleration today if $\mu \sim H_0$.

Besides the final attractor, a coupled field with an exponential potential has also another solution in which matter and field scale in the same way with time, and, consequently, their density fractions are constants. This epoch has been denoted as the ϕ-matter-dominated era (ϕMDE) [11]. As we will show in a moment, the ϕMDE plays a central role in this work. This epoch occurs just after the radiation era and replaces the usual MDE. During the ϕMDE, the energy fraction $\tilde{\Omega}_\phi$ and the effective EOS $\tilde{w}_{\rm eff}$ are constant and given by [11,12] $\tilde{\Omega}_\phi = \tilde{w}_{\rm eff} = 4\beta^2/9$. Then we have $\tilde{\Omega}_\phi = \tilde{w}_{\rm eff} = 1/9$ in $f(R)$ modified gravity theories. Therefore, contrary to standard cosmology, in coupled models, DE is not negligible in the past (until the radiation era). In contrast to the accelerated attractor, the ϕMDE occurs when the coupling term in the right-hand side of Eq. (5) dominates over the potential term, as it can be explicitly shown. This aspect is crucial for the present work, since it implies that the ϕMDE exists independently of the form of $f(R)$. In this regime, the scale factor behaves as $\tilde{a} \sim \tilde{t}^{3/5}$. In the JF, this becomes $a \sim t^{1/2}$ instead of the usual $t^{2/3}$ behavior of the MDE. This is clearly a strong deviation from standard cosmology and is ruled out by observations, as illustrated below. Notice that the JF evolution in this phase corresponds to $R = 0$ as during the radiation epoch, but, just as in that case, there is no singularity in an inverse power-law theory because this behavior is not exact (see below). In the language of dynamical systems, the ϕMDE is a saddle point.

To analyze the full system (including the radiation energy density $\rho_{\rm rad}$, which obeys the standard conservation equation), we introduce the following quantities:

$$x_1 = \frac{\kappa\phi'}{\sqrt{6}}, \qquad x_2 = \frac{\kappa}{\tilde{H}}\sqrt{\frac{V}{3}}, \qquad x_3 = \frac{\kappa}{\tilde{H}}\sqrt{\frac{\rho_{\rm rad}}{3}}, \tag{9}$$

where a prime denotes the derivative with respect to $N \equiv \log(\tilde{a})$. The energy fractions of the field ϕ and of matter are given by $\tilde{\Omega}_\phi = x_1^2 + x_2^2$ and $\tilde{\Omega}_m = 1 - x_1^2 - x_2^2 - x_3^2$, respectively. The effective EOS and the field EOS are $\tilde{w}_{\rm eff} = x_1^2 - x_2^2 + x_3^2/3$ and $\tilde{w}_\phi = (x_1^2 - x_2^2)/(x_1^2 + x_2^2)$, respectively. The complete system has been already studied in Ref. [11], and we will not repeat it here. The ϕMDE corresponds to the fixed points $(x_1, x_2, x_3) = (1/3, 0, 0)$, with $\tilde{\Omega}_\phi = \tilde{w}_{\rm eff} = 1/9$. After this, the Universe falls on the final attractor, which is the accelerated fixed point $(x_1, x_2, x_3) = (\lambda/\sqrt{6}, \sqrt{1-\lambda^2/6}, 0)$, with $\tilde{\Omega}_\phi = 1$ and $\tilde{w}_{\rm eff} = -1 + \lambda^2/3$.

To return to the JF, we can simply apply the transformation law (4). In the regime where radiation is negligible ($x_3 \simeq 0$), we obtain the following effective EOS:

$$w_{\rm eff} = \frac{1}{3} - \frac{2n}{3(1+n)}\frac{x_2^2}{(1-x_1)^2}, \tag{10}$$

together with the relation

$$\Omega_m \equiv \frac{\kappa^2\rho_m}{3H^2F} = \frac{\tilde{\Omega}_m}{(1-x_1)^2}. \tag{11}$$

For the accelerated attractor, we have $w_{\rm eff} = -1 + [2(2+n)/3(1+n)(1+2n)]$ (this relation was originally found in the context of inflation in Ref. [13]), which gives $w_{\rm eff} = -2/3$ for $n = 1$. The ϕMDE corresponds to $x_2 = 0$ and, therefore, $w_{\rm eff} = 1/3$ for any n, giving $a \propto t^{1/2}$. From Eq. (11), one has $\Omega_m = 2$ and $\Omega_R \equiv (RF - f - 6H\dot{F})/6FH^2 = -1$ in the ϕMDE [see Eq. (2)]. Since Ω_R does not need to be positive-definite, Ω_m can be larger than unity without any inconsistency.

Notice also that w_ϕ differs from the quantity $w_{\rm DE}$ used to analyze SN data and defined through the equation $H^2 = H_0^2[\Omega_m^{(0)}a^{-3} + (1-\Omega_m^{(0)})a^{-3}\exp(-3\int da w_{\rm DE}/a)]$. We will also computte $w_{\rm DE}$ below.

Most modifications of gravity suggested in the literature consider terms in addition to the usual Einstein-Hilbert Lagrangian. For instance, several authors have studied the following DE model [2,3]:

$$f(R) = R - \mu^{2(n+1)}/R^n. \tag{12}$$

In this case, the potential in the EF is given by

$$V(\phi) = Ae^{-(2\sqrt{6}/3)\kappa\phi}(e^{(\sqrt{6}/3)\kappa\phi} - 1)^{n/n+1}, \tag{13}$$

which vanishes at $\phi = 0$ and has a maximum at $\kappa\phi = 2(n+1)/(n+2)$. In the limit $\phi \to \infty$, it behaves as $V(\phi) \propto \exp(-\lambda\kappa\phi)$. For negative n, the pure exponential approximation is always good during the past cosmic history if the higher-curvature term is responsible for the present acceleration, because we are always in the large R limit. For positive n, the potential (13) differs from the pure exponential one in the limit $R \gg \mu^2$, and one might expect that for these values the evolution goes on as in the standard case and, in particular, the ϕMDE disappears.

However, we show now by building an explicit solution that in reality this does not happen.

Let us focus on the $R - \mu^4/R$ model taken for simplicity without radiation. During the ϕMDE, one can approximate the scale factor in the JF as $a(t) = (t/t_i)^{1/2} + \epsilon(t)(t/t_i)^{9/4}$, where t_i is an initial time at the beginning of the ϕMDE. This solution is valid at first order in ϵ provided $\epsilon = (\mu^2/144H_i^2)/[(\rho_m^{(i)}/3H_i^2) - \sqrt{H/H_i}]^{1/2}$, which is indeed small for μ of order H_0 as present acceleration requires. After some time, the correction gets larger than the zeroth order term, and the ϕMDE is followed by a phase of accelerated expansion. Then the beginning of the late-time acceleration is quantified by the condition $t \approx [144H_i^2\sqrt{(\rho_m^{(i)}/3H_i^2)}/\mu^2]^{4/7}t_i$. A similar argument applies for any $n < -1$, $n > -3/4$ with a correction growing as $t^{5/2-1/2(n+1)}$. Since R is of order μ^2 at the beginning of the ϕMDE, μ^4/R dominates over R after the radiation era. This applies for any μ, no matter how small. In other words, the limit $\mu \to 0$ of a fourth-order theory does not reduce to second-order general relativity if, at the same time, one imposes the conditions of acceleration today (cf. [14]).

For larger μ ($\gg H_0$), the ϕMDE can be shortened or bypassed from the above condition, but then DE dominates soon without a matter-dominated epoch. Thus, we have only two cases: either (i) the ϕMDE exists, or (ii) a rapid transition from the radiation era to the accelerating stage (without ϕMDE) takes place. In summary, the system never behaves as in a standard cosmological scenario except during radiation (during which matter and field play no role in the expansion rate). In other words, whenever matter is dynamically important, DE is also important as a consequence of the coupling. We now confirm all of this by a direct numerical integration.

In Fig. 1, we plot the evolution of Ω_ϕ, $\Omega_{\rm DM}$, $\Omega_{\rm rad}$, and the equation of state in the EF for $n = 1$, 4, and 10 and $\mu \approx H_0$. The present values of the radiation and matter density fractions are chosen to match the observations in the JF. As expected, the system enters the ϕMDE after the radiation era and finally falls on the accelerated attractor. We ran our numerical code for other positive and negative values of n (from $n = -10$ to $n = 10$, limiting to the accelerating cases) and found similar cosmological evolutions. The plots in Fig. 1 are therefore qualitatively valid for any $R + \alpha R^{-n}$ model (provided $F > 0$). It is also interesting to observe the fast variation of the EOS near $z = 0$; this shows that a linear parametrization of $w_{\rm DE}$ is useful only for a limited redshift range.

We can show now that an effective EOS $w_{\rm eff} = 1/3$ during the ϕMDE is cosmologically unacceptable. In principle, this should be shown case by case by a complete likelihood analysis of CMB and LSS data (see [15] for such an analysis for various coupled models), but this program is hardly feasible if we want to make general statements on $f(R)$ theories. Instead, we take a simpler but general approach. We calculate the angular size of the sound horizon

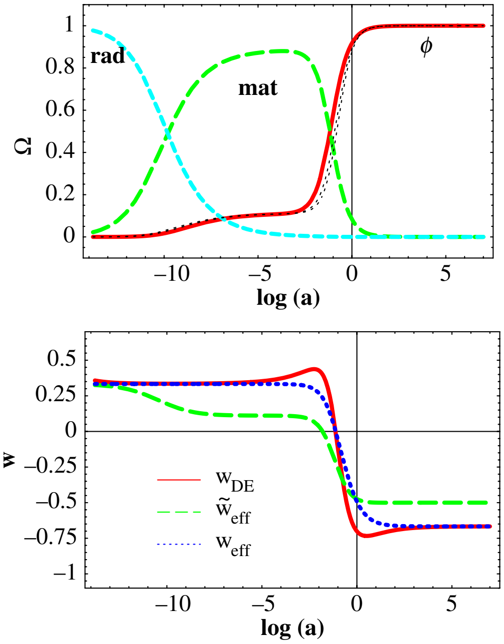

FIG. 1 (color online). Evolution of the fractional energy densities (ϕ, matter, radiation) in the EF for the model $f(R) = R - \mu^4/R$ (top panel). Superimposed as dotted lines is the evolution of $\tilde{\Omega}_\phi$ for $n = 4$ and 10. Notice the constant value $\tilde{\Omega}_\phi \simeq 1/9$ in the ϕMDE phase between radiation and DE domination. In the bottom panel, we plot the evolution of the observed EOS $w_{\rm DE}$ of DE in the JF and the effective EOS in both the EF and the JF ($n = 1$).

$$\theta_s = \int_{z_{\rm dec}}^{\infty} \frac{c_s(z)dz}{H(z)} \Bigg/ \int_0^{z_{\rm dec}} \frac{dz}{H(z)}, \tag{14}$$

where $c_s^2(z) = 1/[3(1 + 3\rho_b/4\rho_\gamma)]$ is the adiabatic baryon-photon sound speed. According to the WMAP3y (Wilkinson Microwave Anisotropy Probe 3 year) results [16], the currently measured value assuming a constant w is $\theta_s = 0.5946 \pm 0.0021°$. As radiation follows the same conservation law, the thermal history is the same as in usual cosmology so that $z_{\rm dec}$ is unchanged. It is easy to show that the integrand $dz/H(z)$ is conformally invariant. For the model (12), we integrate numerically the equations of motion in the EF (including radiation) by changing initial conditions for x_1, x_2, and x_3 via a trial and error procedure until we obtain the present Universe with JF matter and radiation densities as observed in the WMAP data (we used $\Omega_m^{(0)} = 0.3$ and $h = 0.7$). Once we have the full background solution, we evaluate θ_s with $z_{\rm dec}$ obtained by solving the relation $\tilde{z}_{\rm dec}e^{\kappa(\phi-\phi_0)/\sqrt{6}} = z_{\rm dec}$. We have two competing effects: First, since the ϕMDE is more decelerated than in the usual case, $r(z_{\rm dec})$ will be systematically smaller; second, the physical sound horizon distance at decoupling is smaller than in usual cosmology, partly because in our models $a \propto t^{1/2}$ also between $t_{\rm eq}$ and $t_{\rm dec}$ with $z_{\rm eq} > z_{\rm dec}$ and mostly because $H(z)$ before decoupling is much higher than in the standard case. Assuming

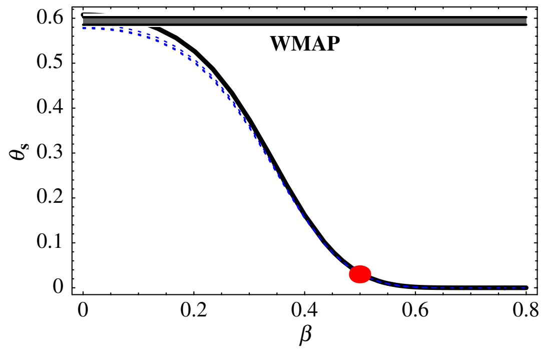

FIG. 2 (color online). The sound horizon angular distance θ_s as a function of the coupling β for $n = 1$ (thick line) and $n = -2$, 3, and 10 (dotted lines). The disk marks the value for $\beta = 1/2$. The gray region shows the WMAP3y constraint at 4σ.

$H(z) = H_0(1 + z)^2$ instead of $(1 + z)^{3/2}$, $H(z_{\rm dec})$ becomes 30 times larger than in standard models. We find that the second effect is by far the dominating one, and as a consequence θ_s turns out to be an order of magnitude smaller than the observed value. In practice, we find that θ_s can be approximated to a few percent using a standard cosmological model with an uncoupled dark energy component and a matter component with an effective equation of state $w = 1/3$. The typical values we find are near $\theta_s \approx 0.03°$, i.e., more than 10 times smaller than in a standard model. The periodic spacing $\Delta\ell$ between the acoustic peaks in the CMB will be larger too by nearly the same factor.

In Fig. 2, we plot the value of θ_s as a function of the coupling constant β [see Eqs. (5) and (6)] and for several n's. Clearly, there is no way that small changes in $\Omega_m^{(0)}$, h, or $w_{\rm eff}$ can cure this problem. Discrepancies are found as well for the age of the Universe, which turns out to be near 10–11 Gyr. As expected, we also find that the perturbations depart significantly from the standard case (as in pure exponential coupled models [15]): For the matter density contrast on subhorizon scales, we find $\delta \sim a^2$ during ϕMDE instead of the standard linear law.

It is possible to generalize our results in several ways. First, one can show by direct substitution that the standard matter era $t^{2/3}$ is a solution of (2) only for pure power laws (plus possibly a cosmological constant) R^{-n}, with $n = -1$, $-(7 \pm \sqrt{73})/12$. In the last two cases, however, the "matter" era occurs for $\Omega_m = 0$ and is, therefore, unacceptable. The first case is clearly the pure Einstein case: This shows that a standard sequence of (exact) $t^{2/3}$ expansion followed by acceleration can occur only for ΛCDM. In contrast, the ϕMDE generically exists as a saddle point. For R^{-n}, with $-1 < n < -3/4$, the ϕMDE is instead a stable point, and the models are ruled out anyway. Still, this alone does not guarantee that the ϕMDE is always reached, regardless of the initial conditions, and a case-by-case numerical analysis is necessary. We explored extensively models such as $f(R) = \exp(R)$ and $\log(R)$ and always found ϕMDE before acceleration. We also carried out a preliminary analysis of Lagrangians for the models such as $f = R - c(R_{\mu\nu}R^{\mu\nu})^{-n}$, $f = R - c(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})^{-n}$, and $f = R - c(R_{\rm GB})^{-n}$ (where $R_{\rm GB}$ is a Gauss-Bonnet term) and did not find any acceptable cosmological evolution.

For the $f(R) = R - \mu_1^4/R + \mu_2 R^2$ models, two mechanisms that could satisfy the local gravity constraints were suggested. One [6] achieves a short interaction range (or a large field mass) by adjusting μ_1 and μ_2 so that $d^2V(\phi)/d\phi^2$ vanishes today, when $R = \sqrt{3}\mu_1^2$. Before this, the R^2 term dominates, and, therefore, we are back in one of our cases and the ϕMDE takes place. Moreover, we find that the local minimum in the EF potential for such a class of Lagrangians does not lead to a late-time (effective) cosmological constant. Another possibility to build a large mass is to take a very small μ_2 [7], but in this case it is the $1/R$ term that dominates the cosmological evolution from the end of radiation, and again we are back in one of our cases. So even models that are designed to pass local gravity experiments fail our cosmological test. In summary, the main feature of our analysis is the modification of the standard matter-dominated epoch for the $f(R)$ dark energy models investigated here. Hence, these models are ruled out as viable cosmologies even if they are arranged to pass the supernovae test and the local gravity constraints. We conjecture that our results apply to a much larger class of $f(R)$ models; the precise conditions that determine the cosmological behavior will be published in future works.

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[1] V. Sahni and A. A. Starobinsky, Int. J. Mod. Phys. D **9**, 373 (2000); T. Padmanabhan, Phys. Rep. **380**, 235 (2003); E. J. Copeland *et al.*, Int. J. Mod. Phys. D **15**, 1753 (2006).
[2] S. Capozziello *et al.*, Int. J. Mod. Phys. D **12**, 1969 (2003).
[3] S. M. Carroll *et al.*, Phys. Rev. D **70**, 043528 (2004).
[4] A. D. Dolgov and M. Kawasaki, Phys. Lett. B **573**, 1 (2003); T. Chiba, Phys. Lett. B **575**, 1 (2003).
[5] P. Teyssandier and P. Tourrenc, J. Math. Phys. (N.Y.) **24**, 2793 (1983).
[6] S. Nojiri and S. D. Odintsov, Phys. Rev. D **68**, 123512 (2003).
[7] A. W. Brookfield, C. van de Bruck, and L. M. H. Hall, Phys. Rev. D **74**, 064028 (2006).
[8] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[9] B. Boisseau *et al.*, Phys. Rev. Lett. **85**, 2236 (2000).
[10] G. Esposito-Farèse and D. Polarski, Phys. Rev. D **63**, 063504 (2001).
[11] L. Amendola, Phys. Rev. D **62**, 043511 (2000).
[12] B. Gumjudpai *et al.*, J. Cosmol. Astropart. Phys. 06 (2005) 007; S. Tsujikawa, Phys. Rev. D **73**, 103504 (2006).
[13] S. Capozziello, F. Occhionero, and L. Amendola, Int. J. Mod. Phys. D **1**, 615 (1992).
[14] V. Faraoni, Phys. Rev. D **74**, 023529 (2006).
[15] L. Amendola and C. Quercellini, Phys. Rev. D **68**, 023514 (2003).
[16] D. N. Spergel *et al.*, astro-ph/0603449.