Collective Working Regimes for Coupled Heat Engines

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Arrays of coupled heat engines are proposed as a paradigmatic model to study the trade-off between individual and collective behavior in linear irreversible thermodynamics. The analysis reveals the existence of a control parameter which selects different operation regimes of the whole array. In particular, the regimes of maximum efficiency and maximum power are considered, giving for the latter a general derivation of the Curzon-Ahlborn efficiency which surprisingly does not depend on whether or not the individual engines in the array work at maximum power.

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Numerous problems arise in natural and social sciences where a variety of agents must cooperate to accomplish some task: engines, organelles within the living cell, division of labor within an insect colony, or assembly line workers in a factory, to name but a few. One fundamental problem in analyzing such systems is to assess how the specific agents' behaviors affect the overall performance. In the case of heat engines, the thermodynamic efficiency has been one of the most popular performance criteria after Carnot [1]. He found that any engine extracting heat from a reservoir at temperature T_2 has to deliver some heat to a reservoir at lower temperature T_1 to do work. Moreover, Carnot showed that the maximum efficiency in the conversion is $\eta_C = 1 - T_1/T_2$, known as the Carnot efficiency. However this efficiency has little practical relevance, since it refers to processes cycling along reversible paths which deliver work infinitely slowly. The limitations of equilibrium thermodynamics to formulate useful criteria describing the performance of real engines have motivated the development of a new field known as finite-time thermodynamics (FTT) [2,3], which, while keeping the formalism as close as possible to that of equilibrium thermodynamics, introduces simple modifications to take into account the main sources of irreversibility observed in real engines.

A paradigmatic model in FTT is due to Curzon and Ahlborn (CA) [4] (see also [5]), who considered a Carnot cycle in finite time and in the so-called endoreversible approximation; i.e., the only sources of irreversibility are associated with the heat transfers between the reservoirs and the working system. Assuming that the heat transfers obey a Fourier law, they found that the efficiency at maximum power of the engine is given by

$$\eta_{\rm CA} = 1 - \sqrt{T_1/T_2}. \tag{1}$$

This result, though subject to some controversy [6], has been recently derived from the theory of linear irreversible thermodynamics in systems of coupled heat engines under the assumption that they all work at maximum power [7]. This is a salient feature because it opens the possibility to analyze nonisothermal heat engines and refrigerators within the framework of linear irreversible thermodynamics [8]; a field up to now almost limited to isothermal energy converters [9].

A striking fact about Eq. (1) is that it provides a surprisingly good approximation to the observed efficiencies of very different power plants [2,3,10,11], suggesting that it represents an "universal" behavior rather than a model specific feature. The main purpose of this Letter is to address the issue of the relationship between overall and individual behaviors in systems of coupled heat engines. Our analytical study reveals that such a relationship is far from obvious, and specifically we will show that the derivation in [7] is a particular case of a much more general result which could shed some light on the reasons why Eq. (1) approaches so well the observed efficiencies of some real engines.

Our starting point is the same construction as in [7]: an array of coupled heat engines, each working between different auxiliary reservoirs (Fig. 1). For simplicity we assume that the temperature profile strictly increases from

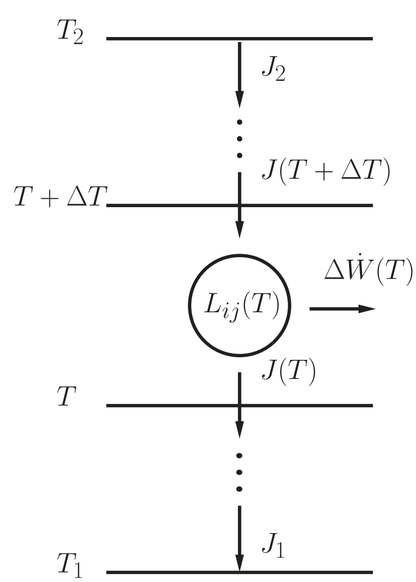

FIG. 1. Array of coupled heat engines.

T_1 to T_2, so that we can use the temperature T to label each engine. These heat engines are coupled in the following sense: the heat output from the engine element at temperature $T + \Delta T$ is exactly equal to the heat input to the next engine element at temperature T; hence, the complete array can be considered as a single engine whose overall behavior is determined by the heat exchanges with the reservoirs at temperatures T_1 and T_2. To see this, consider the elemental engine working between temperatures T and $T + \Delta T$ (Fig. 1): heat is allowed to flow at a rate $J(T + \Delta T)$ from the hot reservoir at temperature $T + \Delta T$ to the cold reservoir at temperature T, while part of the heat is turned into work delivered to the surroundings at a rate $\Delta \dot{W}(T)$. This work is done against an external force $f(T)\Delta T$ with conjugate variable $x(T)$ and flux $v(T) = \dot{x}(T)$, hence $\Delta \dot{W}(T) = f(T)v(T)\Delta T$. Conservation of energy in this engine implies that $J(T + \Delta T) = J(T) + f(T)v(T)\Delta T$, thus in the limit $\Delta T \to 0$ we get

$$J'(T) = f(T)v(T), \tag{2}$$

where the prime denotes differentiation with respect to T. Integrating this equation gives the net power delivered by the total array in terms of the heat fluxes $J_1 = J(T_1)$, $J_2 = J(T_2)$ at the colder and hotter reservoirs:

$$\dot{W} = \int_{T_1}^{T_2} dT f(T)v(T) = J_2 - J_1. \tag{3}$$

Therefore, the efficiency of this energy conversion is

$$\eta = \frac{\dot{W}}{J_2} = 1 - \frac{J_1}{J_2}. \tag{4}$$

The rate of entropy production due to the elemental engine operating between reservoirs T and $T + \Delta T$ can be simply written as $\Delta \dot{S}(T) = J(T + \Delta T)/(T + \Delta T) - J(T)/T$. Taking into account the conservation of energy we get at first order in ΔT

$$\Delta \dot{S}(T) = \left(-\frac{J(T)}{T^2} + \frac{f(T)v(T)}{T}\right)\Delta T. \tag{5}$$

With the help of Eq. (2) this last expression implies in the limit $\Delta T \to 0$ that $\dot{S}'(T) = (J(T)/T)'$. Integrating from T_1 to T_2 gives the total rate of entropy production,

$$\dot{S} = \frac{J_2}{T_2} - \frac{J_1}{T_1}. \tag{6}$$

Expression (5) suggests considering $(1/T)' = -1/T^2$ and $f(T)/T$ as thermodynamic forces with conjugate fluxes $J(T)$ and $v(T)$, respectively. For small values of the thermodynamic forces and under the assumption of local equilibrium, linear irreversible thermodynamics [12] allows us to write the following relationships:

$$v(T) = L_{11}(T)\frac{f(T)}{T} - L_{12}(T)\frac{1}{T^2}, \tag{7}$$

$$J(T) = L_{21}(T)\frac{f(T)}{T} - L_{22}(T)\frac{1}{T^2}, \tag{8}$$

where the Onsager coefficients $L_{ij}(T)$ satisfy

$$L_{11}(T) \geq 0, \qquad L_{22}(T) \geq 0, \qquad L_{12}(T) = L_{21}(T), \tag{9}$$

$$L_{11}(T)L_{22}(T) - L_{12}(T)^2 \geq 0. \tag{10}$$

These properties ensure the positivity of Eq. (5) and therefore that of the total entropy production Eq. (6). As a consequence, Eq. (4) is bounded from above by the Carnot efficiency, which is reached when $\dot{S} = 0$.

It should be noticed that once the temperatures T_1, T_2 and the Onsager coefficients $L_{ij}(T)$ are fixed, the force profile $f(T)$ cannot be chosen freely. The reason is that conservation of energy in every engine expressed by Eq. (2) together with the transport Eqs. (7) and (8) implies a Ricatti differential equation for $f(T)$:

$$[L_{12}(T)f(T)]' = L_{11}(T)f(T)^2 + T\left(\frac{L_{22}(T)}{T^2}\right)'. \tag{11}$$

In the most general case there is no way of writing the general solution of Eq. (11) by using some quadratures. However, one can integrate it completely if some extra information is added. For example, if one particular solution of Eq. (11) is known, the problem can be reduced to an inhomogeneous first order linear equation and the general solution can be found by two quadratures [13].

There exists an interesting class of functions $L_{ij}(T)$ which allows us to find such a particular solution quite easily by physical arguments. Let us introduce the so-called coupling strength parameter

$$q(T) = \frac{L_{12}(T)}{\sqrt{L_{11}(T)L_{22}(T)}}, \tag{12}$$

which measures the degree of coupling between the fluxes $v(T)$ and $J(T)$. Because of the properties (9) and (10) of the Onsager coefficients this parameter obeys $-1 \leq q(T) \leq +1$. Now assume that all the elemental engines in the construction satisfy $|q(T)| = 1$. In such a case $v(T)$ and $J(T)$ are tightly coupled for all T and if one of them vanishes the other flux must vanish also. Imposing $v(T) = 0$, $J(T) = 0$ we get

$$f_{\mathrm{eq}}(T) = \frac{L_{22}(T)}{TL_{12}(T)} = \frac{L_{12}(T)}{TL_{11}(T)}, \tag{13}$$

which simply represents a force profile that ensures an equilibrium state in which the fluxes (7) and (8) and the entropy production (6) vanish. It is easy to check that Eq. (13) satisfies the differential Eq. (11) when $|q(T)| = 1$. In such a case its general solution can be constructed by standard methods [13]. The result is

$$f(T) = f_{\mathrm{eq}}(T) - \frac{\lambda T^2}{L_{12}(T)[1 + \lambda h(T)]}, \tag{14}$$

where λ is an integration constant and $h(T)$ is a positive, increasing function of T:

$$h(T) = \int_{T_1}^{T} dT' \frac{T'^2}{L_{22}(T')}. \quad (15)$$

Clearly, in order to avoid divergences in (14), it is necessary that $\lambda > -1/h(T_2)$, otherwise the linear relationships (7) and (8) would be difficult to justify. Once the general form of the force profile (14) is known we can proceed to calculate the relevant thermodynamic quantities. The heat flux (8) crossing each reservoir is simply

$$J(T) = -\lambda T/[1 + \lambda h(T)] \quad (16)$$

and the total power (3) delivered by the array is

$$\dot{W}(\lambda) = J_2(\lambda) - J_1(\lambda) = -\frac{\lambda T_2}{1 + \lambda h(T_2)} + \lambda T_1. \quad (17)$$

At this point it is apparent that the parameter λ determines which kind of energy converter we have. A closer inspection of Eqs. (16) and (17) shows that the system works as a refrigerator [$J(T) > 0$, i.e., heat flows up the temperature gradient and the power $\dot{W} > 0$ is consumed by the array] for $\lambda \in (-1/h(T_2), 0)$, while it may work as a motor [$J(T) < 0$ and $\dot{W} < 0$] only for $\lambda > 0$ [Fig. 2(a)]. In such a case, the efficiency (4) is given by

$$\eta(\lambda) = 1 - [1 + \lambda h(T_2)]\frac{T_1}{T_2}. \quad (18)$$

We furthermore notice that λ can also be viewed as a control parameter which allows us to select different operation regimes of the array. For instance, to find the regime of maximum efficiency usually one has to solve the equation $\partial\eta(\lambda)/\partial\lambda = 0$ for λ. However, when $|q(T)| = 1$ Eq. (18) is a decreasing function of λ so that the maximum efficiency is reached at the lower limit of the motor interval ($\lambda = 0$) and coincides with the Carnot efficiency. Such a result is not surprising, since for $\lambda = 0$ we have $f(T) = f_{\rm eq}(T)$, so that every elemental engine in the array is in equilibrium and works with an efficiency given by the Carnot value $\eta(T) = \Delta T/T$. Because of the invariance of the Carnot efficiency under the coupling of engines [7], the whole array amounts to a single equilibrium engine working between the temperatures T_1 and T_2 with $\dot{W} = 0$ and its efficiency is given by the Carnot value. We mention that due to the reversible behavior of engines near equilibrium, the sign of the heat fluxes and the power changes upon variation of λ in a neighborhood of $\lambda = 0$ [Fig. 2(a)].

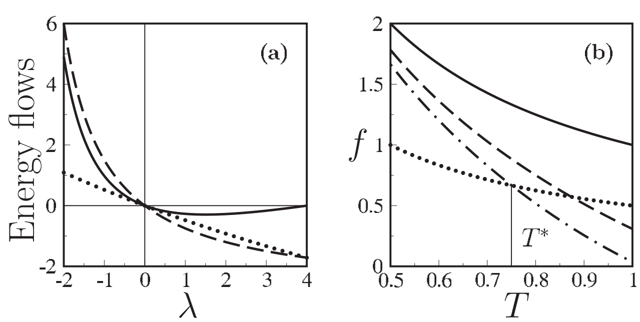

FIG. 2. (a) Plot of $\dot{W}$ (solid line), $J_1 = J(T_1)$ (dotted line), and $J_2 = J(T_2)$ (dashed line) vs. λ [Eqs. (15)–(17)] in a model with constant Onsager coefficients. The parameters are (in arbitrary units) $L_{ij} = 1$, $T_1 = 0.5$, and $T_2 = 1$. (b) Force profiles at maximum efficiency $f_{\rm eq}(T)$ [Eq. (13), solid line], maximum $\Omega = 2\dot{W} - (1 - T_1/T_2)J_2$ (dashed line), and maximum power [Eqs. (14) and (19), dot-dashed line] for the previous model. The dotted lined represents the force profile $f(T) = f_{\rm eq}(T)/2$ [7] which follows from the power maximization of each elemental engine individually.

Regarding the regime of maximum power, we will show in the following that the Curzon-Ahlborn efficiency (1) is a fundamental result in linear irreversible thermodynamics with no other restrictions imposed on the Onsager coefficients than the coupling strength parameter (12) satisfies $|q(T)| = 1$. Solving the equation $\partial\dot{W}(\lambda)/\partial\lambda = 0$ we get after some algebra a second order equation with two roots of opposite sign. The negative root must be discarded, since it lies outside the allowed interval $(-1/h(T_2), +\infty)$. Substituting the positive root

$$\lambda = \frac{\sqrt{T_2} - \sqrt{T_1}}{h(T_2)\sqrt{T_1}} \quad (19)$$

in Eq. (17) gives the maximal power output $\dot{W} = -h(T_2)^{-1}(\sqrt{T_2} - \sqrt{T_1})^2$, which mimics familiar results in FTT, with the factor $h(T_2)^{-1}$ playing the role of an averaged thermal conductance between T_1 and T_2 [2,3,10]. Finally, Eqs. (18) and (19) give the Curzon-Ahlborn result for the efficiency at maximum power: $\eta_{\rm CA} = 1 - \sqrt{T_1/T_2}$.

A different approach to derive Eq. (1) was proposed in [7] which takes advantage of the fact that the Curzon-Ahlborn and Carnot efficiencies share the same invariance property under the coupling of engines. Therefore, if we could couple elemental engines working at maximum power with an efficiency given by Eq. (1) the result would be an engine which already works at maximum power with the Curzon-Ahlborn efficiency. We emphasize that this can be done only for a limited class of systems. The reason is as follows: maximizing the power of every elemental engine in the array we obtain $f(T) = f_{\rm eq}(T)/2$ [7], and substituting this force profile in Eq. (11) gives a differential equation which amounts to an additional restriction imposed on the Onsager coefficients. With the help of Eq. (12) and assuming that $|q(T)| = q$ for all T, it can be written as

$$\left(\frac{L_{22}(T)}{T}\right)' = \left(\frac{4 - q^2}{4 - 2q^2}\right)\frac{L_{22}(T)}{T^2}. \quad (20)$$

The general solution is

$$L_{22}(T) = CT^{(4-q^2)/(4-2q^2)}, \quad (21)$$

where C is an integration constant. Then Eq. (21) defines the class of systems tacitly assumed in the derivation of the

Curzon-Ahlborn efficiency in [7]: only these engines can be coupled working individually at maximum power.

In conclusion, the global optimization of the total power $\dot{W}(\lambda)$ with respect to λ shows that, in contrast to the derivation in [7], the Curzon-Ahlborn efficiency does not necessarily require that every unit performs work at maximum power, thus Eq. (1) has a wider scope than so far was expected. This idea is apparent in Fig. 2(b), which shows force profiles for the maximum efficiency and maximum power regimes of the array in a specific model with T-independent Onsager coefficients: while in the regime of maximum efficiency the force profile given by Eq. (13) implies that every elemental engine is in equilibrium and works with the Carnot efficiency, at maximum power a single elemental engine at temperature $T^* \in (T_1, T_2)$ works in such operation regime. It seems plausible to argue that a real power plant could display an efficiency close to the Curzon-Ahlborn value if the processes taking place in it are strongly coupled and the global performance is optimized to give the maximum power. Moreover, the improvements carried out along the last decades in the design of different power plants and in its components has witnessed a concomitant convergence of the observed efficiencies towards the values given by Eq. (1).

Besides power and efficiency, one can consider different figures of merit. Since "one cannot have it all" [14] some compromise-based criteria have been proposed [3]. Here we consider the so-called Ω criterion [15], it represents a trade-off between useful energy delivered to the surroundings and energy lost by any energy converter which is specially easy to implement. In our case and for $|q(T)| = 1$ it can be written as $\Omega = 2\dot{W} - (1-\tau)J_2$, where $\tau = T_1/T_2$. The force profile which maximizes Ω [Fig. 2(b)] can be found in the same way as already done for efficiency and power, again it differs from the one maximizing the Ω-function for each unit $(3f_{\mathrm{eq}}(T)/4)$. The efficiency in such operation regime is $\eta_\Omega = 1 - \sqrt{\tau(1+\tau)/2}$, a well-known result of some endoreversible models in FTT [3,15].

The previous results suggest that, given the constraints inherent in the couplings, it is not necessary (nor generally possible) to impose the same operation regime to every agent in order to achieve a desired overall performance. It shows also that when $|q(T)| = 1$ some performance criteria, as the efficiency, may become independent of the structure of the system, as detailed by the functions $L_{ij}(T)$, in some operation regimes (maximum efficiency, maximum power or maximum Ω).

Unfortunately, we have not found a general solution of Eq. (11) for arbitrary q. Nevertheless we have analyzed various models for which Eq. (11) has particular solutions of the form $f(T) \propto 1/T$ and compared the results which stem from the previous theory with those coming from the FTT formalism using the standard Carnot-like models incorporating both external and internal irreversibilities. When the array works as a motor [Fig. 2(a), $\lambda > 0$] the parametric plot efficiency versus power for $|q| = 1$ is an open curve typical of the endoreversible Carnot-like models, while for $|q| < 1$ the plot becomes loop-shaped, in agreement with results of real heat engines and with predictions of irreversible models in FTT [2,3]. Finally, in the case of refrigerators, Fig. 2(a) ($\lambda < 0$) suggests that there is no obvious thermodynamic function which could be optimized, though such functions could exist for particular choices of the Onsager coefficients, see [8].

In summary, we have developed a theory which allows a detailed analysis within linear irreversible thermodynamics of arrays of coupled heat engines working between arbitrary temperature differences. A control parameter has been found which selects different operation regimes for the whole array without imposing the same regime to every engine in the array. In the limit of tightly coupled fluxes ($|q| = 1$) we showed that the efficiency becomes a function of $\tau = T_1/T_2$ which is independent of the structure of the system [i.e., of $L_{ij}(T)$] in various operation regimes, as occurs in endoreversible models in FTT, while for $|q| < 1$ the qualitative behavior of irreversible models is recovered.

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