Quantum-Electrodynamical Photon Splitting in Magnetized Nonlinear Pair Plasmas

G. Brodin,[1] M. Marklund,[1] B. Eliasson,[1,2] and P. K. Shukla[1,2]

[1]*Department of Physics, Umeå University, SE-901 87 Umeå, Sweden*

[2]*Institut für Theoretische Physik IV, Ruhr-Universität Bochum, D-44780 Bochum, Germany*

(Received 3 May 2006; published 20 March 2007)

We present for the first time the nonlinear dynamics of quantum electrodynamic (QED) photon splitting in a strongly magnetized electron-positron (pair) plasma. By using a QED corrected Maxwell equation, we derive a set of equations that exhibit nonlinear couplings between electromagnetic (EM) waves due to nonlinear plasma currents and QED polarization and magnetization effects. Numerical analyses of our coupled nonlinear EM wave equations reveal the possibility of a more efficient decay channel, as well as new features of energy exchange among the three EM modes that are nonlinearly interacting in magnetized pair plasmas. Possible applications of our investigation to astrophysical settings, such as magnetars, are pointed out.

DOI: 10.1103/PhysRevLett.98.125001 PACS numbers: 52.35.Mw, 52.27.Ep, 95.30.Gv

Recently, there has been a great deal of interest [1] in the investigation of effects associated with radiation pressure and quantum vacuum fluctuations in nonlinear media. Such studies are of importance in astrophysical environments, where copious amounts of electron-positron pairs exist due to numerous physical processes [1]. Elastic photon-photon scattering is traditionally described within quantum electrodynamic (QED) [1,2]. However, observable effects of elastic photon-photon scattering among real photons have so far not been reported in the laboratory [1,3]. For astrophysical systems [4,5] the situation is different, since the large magnetic field strength in pulsar and magnetar [6] environments changes the diamagnetic properties of vacuum significantly [7], and leads to phenomena such as frequency downshifting [5]. The latter is a result of photon-splitting [8,9], and the process may be responsible for the radio silence of magnetars [5,10]. Moreover, the propagation of electromagnetic waves in a relativistically dense electron gas [11] and in a relativistic electron-positron gas [12] have been discussed, leading to the important effect of gamma photon capture and pair plasma suppression around pulsars [13].

In this Letter, we present the nonlinear photon splitting of electromagnetic (EM) waves propagating perpendicularly to a strong external magnetic field $\mathbf{B}_0$ in a pair plasma. Because of the QED effect [1], a photon in vacuum can decay into a backscattered and a forward scattered photon, where the latter two photons have polarizations perpendicular to that of the original photon [8,9]. Noting that significant pair production [14] occurs in astrophysical settings (viz. in pulsar and magnetar environments), we demonstrate here a novel possibility of a nonlinear decay interaction, due to a competition between QED and plasma nonlinearities. We note that most of previous investigations [15,16], including both QED and plasma effects, have been limited to linear EM wave propagation. Here we derive three dynamical equations with nonlinear couplings between photons with different polarizations. From these coupled mode equations, the QED cross section for photon splitting [8] can be deduced in the limit of zero plasma density. We discuss applications of our results to magnetar atmospheres.

Photon-photon scattering can be described by the Heisenberg-Euler Lagrangian [17,18] $L = \epsilon_0(E^2 - c^2B^2)/2 + \kappa\epsilon_0^2[(E^2 - c^2B^2)^2 + 7(c\mathbf{E}\cdot\mathbf{B})^2]$. Here $\kappa = (\alpha/90\pi)(1/\epsilon_0 E_{\rm crit}^2)$, $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the fine-structure constant, $E_{\rm crit} = m^2c^3/e\hbar \sim 10^{18}$ V/m is the critical field [1], $\hbar$ is the Planck constant, m is the electron mass, ϵ_0 is the vacuum permittivity, and c is the vacuum speed of light. The last two terms in the Heisenberg-Euler Lagrangian represent the effects of the vacuum polarization and magnetization. The QED corrected Maxwell equations can then be written in their classical form using $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$ and $\mathbf{H} = c^2\epsilon_0\mathbf{B} - \mathbf{M}$, where [1] $\mathbf{P} = 2\epsilon_0^2\kappa[2(E^2 - c^2B^2)\mathbf{E} + 7c^2(\mathbf{E}\cdot\mathbf{B})\mathbf{B}]$ and $\mathbf{M} = 2c^2\epsilon_0^2\kappa[-2(E^2 - c^2B^2)\mathbf{B} + 7(\mathbf{E}\cdot\mathbf{B})\mathbf{E}]$, which are valid for $|\mathbf{E}|$, $c|\mathbf{B}| \ll E_{\rm crit}$ and $\omega \ll \omega_e = mc^2/\hbar \approx 8\times 10^{20}$ rad/s.

Next, we study wave propagation perpendicular to an external magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ in an electron-positron plasma, letting all variables depend on (x, t). Assuming that the charge density is negligible, the wave equation for the electric field $\mathbf{E}$ then reads

$$(\partial_t^2 - c^2\partial_x^2)\mathbf{E} = -\epsilon_0^{-1}[\partial_t\mathbf{j} - c^2\hat{\mathbf{x}}\partial_x^2 P_x], \qquad (1)$$

where $\mathbf{j} = \partial_t\mathbf{P} + \nabla\times\mathbf{M} + \sum_{e,p} qn\mathbf{v}$, $|\mathbf{E}|/c$, $|\mathbf{B}| \ll B_0$, $\mathbf{v}$ denotes the average (fluid) velocity, and the sum is over the electron and positron contributions. The latter are determined from the relativistic equation of motion

$$(\partial_t + \mathbf{v}\cdot\nabla)(\gamma\mathbf{v}) = (q/m)(\mathbf{E} + \mathbf{v}\times\mathbf{B}). \qquad (2)$$

We assume that the EM wave frequency ω and the electron (positron) plasma frequency $\omega_{{\rm pe}(p)}$ are much smaller than the magnitude of the electron (or positron) gyrofrequency $|\omega_{{\rm ce}(p)}| = \omega_c = eB_0/m$, relevant for pulsar and magnetar atmospheres [19]. This ordering make charge density oscillations negligible, as the longitudinal motion is given by the $\mathbf{E}\times\mathbf{B}$-drift to leading order. Next, we linearize and Fourier decompose to obtain the dispersion relations for propagation perpendicular to $\hat{\mathbf{z}}$ [20]

$$\omega^2 \approx k^2c^2(1-8\xi), \tag{3a}$$

$$\omega^2 \approx k^2c^2(1-14\xi)+\omega_p^2, \tag{3b}$$

where $\omega_p = (\omega_{\rm pe}^2+\omega_{\rm pp}^2)^{1/2}$ is the plasma frequency of the pair plasma. Here, $\xi = \kappa\epsilon_0 c^2 B_0^2 \ll 1$, and we assume that $\omega_p^2 \ll \omega^2$ [21]. We have omitted the contribution proportional to $\omega^2\omega_p^2/\omega_c^2$ in (3a), which is smaller than the plasma contribution proportional to ω_p^2 in (3b). We note that the EM mode described by (3a) has the electric field perpendicular to $\mathbf{B}_0$ in the $\hat{\mathbf{y}}$ direction (approximately), whereas the EM mode described by (3b) has the electric field parallel to $\mathbf{B}_0$. Henceforth, the two different polarizations will be denoted by the subscripts $\perp$ and $\|$, respectively. In [16,22], the linear effects from the combined QED and plasma effects are discussed in detail.

Next, we represent the EM waves as $\tilde{E}(x,t)\exp(ikx - i\omega t)$ + complex conjugate, and the slowly varying amplitudes are denoted by tilde. Our aim is to investigate photon splitting [8,9], a parametric process where one photon with perpendicular polarization decays into two photons with parallel polarizations. Denoting the latter waves with indices 1 and 2, the energy and momentum conservation relations (matching conditions) are $\omega_\perp = \omega_{1\|}+\omega_{2\|}$ and $k_\perp = k_{1\|}+k_{2\|}$. We point out that in addition to this process, QED allows a decay of the type $\omega_\perp = \omega_{1\perp}+\omega_{2\|}$ [2]. However, the scattering amplitude of this process is suppressed by a factor of the order $\alpha\xi$ [2]. The presence of a plasma may in principle change this ordering and also add new decay channels, but in the strongly magnetized high-frequency regime considered here, ω, $\omega_c \gg \omega_p$ we note that this is not the case. For $\omega_p = 0$, the simultaneous fulfilment of the matching conditions and the Eqs. (3a) and (3b) requires that one of the EM waves is backscattered, i.e., either $k_{1\|}<0$ or $k_{2\|}<0$. Furthermore, $\xi \ll 1$ means that the backscattered EM wave has a much smaller frequency than $\omega_\perp$. On the other hand, for $\omega_p \neq 0$, the matching conditions also allow for both the decay products to be scattered in the forward direction. Next, we divide all quantities into unperturbed and perturbed parts, i.e., $\mathbf{B} = \mathbf{B}_0+\mathbf{B}_1$ where the perturbed part fulfills $|\mathbf{B}_1| \ll |\mathbf{B}_0|$, and similarly for the polarization, magnetization, and density. We then include the resonant second order nonlinear terms from $\mathbf{P}$ and $\mathbf{M}$ in Maxwell's equations (noting that $\mathbf{P} = \mathbf{P}_0+\mathbf{P}_1+\mathbf{P}_2$, where $\mathbf{P}_2$ is second order in the perturbed EM field, etc.) together with the second order terms in (2) (i.e., the Lorentz force and the convective derivative), and the nonlinear current density in the right-hand side of (1). From the continuity equation $\partial_t\rho_1 = -\rho_0\partial_x v_x$ we solve for the density. After straightforward algebra, substituting linear expressions into the nonlinear terms, we obtain our coupled mode equations

$$\partial_t\tilde{E}_\perp + v_{g\perp}\partial_x\tilde{E}_\perp = \omega_\perp C\tilde{E}_{1\|}\tilde{E}_{2\|}/E_{\rm crit}, \tag{4a}$$

$$\partial_t\tilde{E}_{1\|} + v_{g1}\partial_x\tilde{E}_{1\|} = \omega_{1\|} C\tilde{E}_{2\|}^*\tilde{E}_\perp/E_{\rm crit}, \tag{4b}$$

$$\partial_t\tilde{E}_{2\|} + v_{g2}\partial_x\tilde{E}_{2\|} = \omega_{2\|} C\tilde{E}_{1\|}^*\tilde{E}_\perp/E_{\rm crit}, \tag{4c}$$

where $v_{gj} = \partial\omega_j/\partial k_j$ is the group speed (j equals $\perp$, $1\|$ and $2\|$), and the asterisk denotes complex conjugate. The coupling strength is $C = C_{\rm pl}+C_{\rm QED}$, where $C_{\rm pl} = i(\alpha/90\pi\xi)^{1/2}(k_\perp c/\omega_\perp)(\omega_p^2/\omega_{1\|}\omega_{2\|})$ is due to the plasma nonlinearities [23] and $C_{\rm QED} = 2i(\alpha\xi/90\pi)^{1/2}\times[10(k_\perp c/\omega_\perp)+7(k_{1\|}c/\omega_{1\|}+k_{2\|}c/\omega_{2\|})]$ is due to QED nonlinear interactions [24]. During certain conditions and for sufficiently long times, the multiscale expansion behind Eqs. (4) can break down due to the growth of higher order terms. However, we will assume that, due to convective stabilization (see below), such effects are negligible and the evolution is well described by (4).

Let us now discuss the relative importance of the QED and plasma effects in various regimes.

Case 1.—In the regime $\omega_p/\omega_\perp \ll \xi$, the matching conditions, the dispersion relations, and $\xi \ll 1$ give $k_{2\|} \approx -3\xi k_\perp/2$ and $\omega_{2\|} \approx 3\xi\omega_\perp/2$, where the last approximation is valid to first order in ξ, and we have chosen $\omega_{2\|}$ as the low-frequency backscattered wave with $k_{2\|}<0$. We note that the condition $\omega_p/\omega_\perp \ll \xi$ ensures that the QED effect dominates over the plasma effects.

Case 2.—Increasing the plasma density affect the linear properties of the low-frequency EM mode first. However, we note that the expression $\omega_{2\|} = 3\xi\omega_\perp/2$ even holds when the plasma effect dominates over the QED effect in Eq. (3b) for the low-frequency EM mode. Increasing the plasma frequency to the regime $\omega_p/\omega_\perp \sim \xi$, the nonlinear coefficients $C_{\rm QED}$ and $C_{\rm pl}$ become comparable and the matching conditions further imply that $\omega_{2\|} \sim \omega_p \sim \xi\omega_\perp$ [see the note after Eqs. (3)]. Furthermore, we note that for $\omega_p \gtrsim 3\xi\omega_\perp$ we would have forward scattering instead of backscattering of the low-frequency EM mode.

Case 3.—For larger plasma frequency ($\omega_p/\omega_\perp \gg \xi$) the QED contribution to the frequency $\omega_\perp$ may still dominate over the plasma contribution, but $|C_{\rm QED}| \ll |C_{\rm pl}|$, and for the nonlinear wave interaction we can thus omit the QED effect in this regime.

We note that the system (4) has the conserved energy $\mathcal{E} = \int(|E_\perp|^2+|E_{1\|}|^2+|E_{2\|}|^2)dx$ when $\omega_\perp = \omega_{1\|}+\omega_{2\|}$, and that two other constants are $\mathcal{N}_1 = \int(|E_\perp|^2/\omega_\perp - |E_{1\|}|^2/\omega_{1\|})dx$ and $\mathcal{N}_2 = \int(|E_\perp|^2/\omega_\perp - |E_{2\|}|^2/\omega_{2\|})dx$, corresponding to the Manley-Rowe relations. The constants of motion are used as a check of the numerical calculations presented below. We first make a linear stability analysis in the presence of a pump EM wave. Thus, we consider the decay of a homogeneous intense wave $E_\perp = E_{\perp 0}$, where $|E_{\perp 0}| \gg |E_{1\|}|, |E_{2\|}|$, into daughter EM waves $E_{1\|} = \hat{E}_{1\|}\exp(iKx - i\Omega t)$ and $E_{2\|} = \hat{E}_{2\|}\exp(-iKx + i\Omega t)$. From (4), we obtain the nonlinear dispersion relation $(\Omega - v_{g2\|}K)(\Omega - v_{g1\|}K) = -\omega_{1\|}\omega_{2\|}|C|^2|E_{\perp 0}|^2/E_{\rm crit}^2$. From the latter, the growth rate of the daughter EM waves is obtained as $\Gamma = [\omega_{1\|}\omega_{2\|}|C|^2|E_{\perp 0}|^2/E_{\rm crit}^2 - (v_{g1\|} + v_{g2\|})^2K^2/4]^{1/2}$ for wave numbers $K < 2(\omega_{1\|}\omega_{2\|})^{1/2}|CE_{\perp 0}|/|v_{g2\|} + v_{g1\|}|E_{\rm crit}$.

To study the dynamics of intense EM waves in a strong magnetic field, we numerically solve the coupled Eqs. (4), and display the results in Figs. 1–3. We use 1000 grid points to resolve the numerical domain $-10^6 \le xk_\perp \le 10^6$ with periodic boundary conditions, and 20 000 steps to advance the solution in time. A pseudospectral method is used to approximate the spatial derivatives and a 4th-order Runge-Kutta method for the time stepping. In Figs. 1 and 2, the plasma is absent, so that the only EM wave couplings are due to the QED effect. We used $\xi = 0.01$, yielding $k_{1\|} = 1.017k_\perp$, $k_{2\|} = -0.017k_\perp$, $\omega_\perp = 0.959k_\perp c$, $\omega_{1\|} = 0.943k_\perp c$, and $\omega_{2\|} = 0.016k_\perp c$. In Fig. 1, we present the evolution of an initially homogeneous beam of amplitude $\tilde{E}_\perp = 0.02E_{\rm crit}$. Initially, the two daughter waves grow exponentially, followed by a nonlinear oscillatory phase. Figure 2 exhibits the nonlinear dynamics of a localized wave packet. We observe the decay of the EM pulse into a forward scattered wave $E_{1\|}$ and a backscattered wave $E_{2\|}$. In Fig. 3, we show the dynamics of a localized EM pulse when the plasma effect is important. We consider the particular case $\omega_p = 3\xi\omega_\perp$, so that the low-frequency wave $E_{2\|}$ has approximately zero group speed and a frequency $\omega_2 \approx \omega_p$. We thus used $\xi = 0.01$ and $\omega_p = 3\xi\omega_\perp = 0.03\omega_\perp$, yielding $k_{1\|} = k_\perp$, $k_{2\|} \approx 0$, $\omega_\perp = 0.959k_\perp c$, $\omega_{1\|} = 0.929k_\perp c$, and $\omega_{2\|} = \omega_p \approx 0.030k_\perp c$. The energy of the pump $E_\perp$ is transferred to a forward scattered wave $E_{1\|}$ and zero-group speed waves $E_{2\|}$.

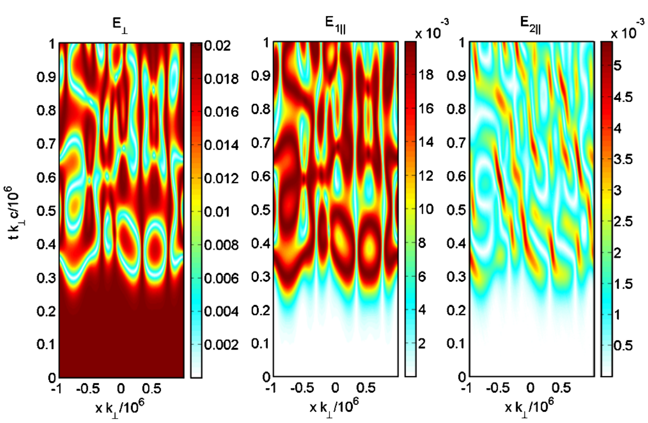

FIG. 1 (color online). The decay of the pump mode $E_\perp = \tilde{E}_\perp/E_{\bf crit}$ into a forward scattered mode $E_{1\|} = \tilde{E}_\perp/E_{\bf crit}$ and a backscattered mode $E_{2\|} = \tilde{E}_\perp/E_{\bf crit}$. Initially, the pump is set to $E_\perp = 0.02$, while E_1 and E_2 is set to a low-level random noise. After the initial exponential decay, the energy is transferred between the pump and the two EM sidebands in a semiperiodic and chaotic manner. We used $\omega_p = 0$ and $\xi = 0.01$, yielding $k_{1\|} = 1.017k_\perp$, $k_{2\|} = -0.017k_\perp$, $\omega_\perp = 0.959k_\perp c$, $\omega_{1\|} = 0.943k_\perp c$, and $\omega_{2\|} = 0.016k_\perp c$.

The present study is of relevance for EM wave propagation in the vicinity of pulsars and magnetars. For example, the radio silence of magnetars is assumed to be connected with the photon splitting in the strong magnetar fields (10^9–10^{11} T) [5,10]. Photon splitting can suppress the creation of electron-positron pairs [10], but we still expect the presence of an electron-positron plasma [14] in such environments. The Goldreich-Julian density is given by [25] $n_{\rm GJ} = 7\times 10^{15}(0.1/\tau)(B_p/10^8)$ m^{-3}, where τ is the pulsar period time (in seconds) and B_p is the surface pulsar magnetic field (in Tesla). The pair plasma density is expected to satisfy $n_e = n_p = Mn_{\rm GJ}$, where a moderate estimate of the multiplicity gives $M = 10$ [26]. Choosing this value and letting $\tau = 1$ s, we note that for the weak magnetar field strength $B_p = 10^9$ T, the characteristic pump frequency $\omega_{\perp\rm char} \sim \omega_p/\xi$, where the QED and plasma effects are of equal importance, has the value $\omega_{\perp\rm char} \sim 4\times 10^{15}$ rad/s, i.e., in the optical range. For $\omega_\perp \ll \omega_{\perp\rm char}$, the plasma nonlinearities dominate, whereas the QED effect dominates in the opposite regime. The evolution of the coupled system of Eqs. (4) is, to a large extent, controlled by the pulse length of the pump mode. For long pulse lengths with $L \gg (\omega_\perp CE_\perp/E_{\rm crit})^{-1}$ (Fig. 1), the system shows a "predator-prey" type of behavior where the energy oscillates chaotically between the modes. For a moderate pulse length with $L \sim (\omega_\perp CE_\perp/E_{\rm crit})^{-1}$, the excited EM wave energy propagates out of the interaction region, and one encounters a more ordered behavior and an effective damping of the pump mode. The EM wave energy is then mainly converted to the parallel polarized forward scattered EM mode (Fig. 2). For a short pulse length with $L \ll (\omega_\perp CE_\perp/E_{\rm crit})^{-1}$, thermal fluctuations do not grow due to convective stabilization (the growth rate Γ vanishes for large wave numbers), and the nonlinear interaction vanishes. As the plasma density

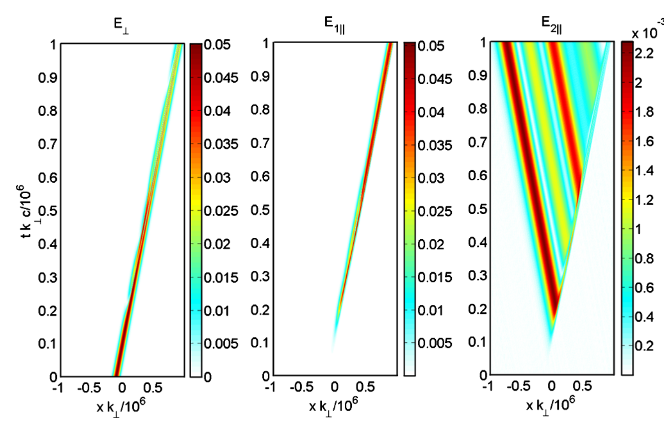

FIG. 2 (color online). The decay of the pump mode $E_\perp = \tilde{E}_\perp/E_{\bf crit}$. Initially the localized pump was set to $E_\perp = 0.05\exp[-(xk_\perp - 10^5)^2/2.5\times 10^9]$. The pump decays into a forward scattered wave $E_{1\|}$ and a backscattered wave $E_{2\|}$. We used $\omega_p = 0$ and $\xi = 0.01$, yielding $k_{1\|} = 1.017k_\perp$, $k_{2\|} = -0.017k_\perp$, $\omega_\perp = 0.959k_\perp c$, $\omega_{1\|} = 0.943k_\perp c$, and $\omega_{2\|} = 0.016k_\perp c$.

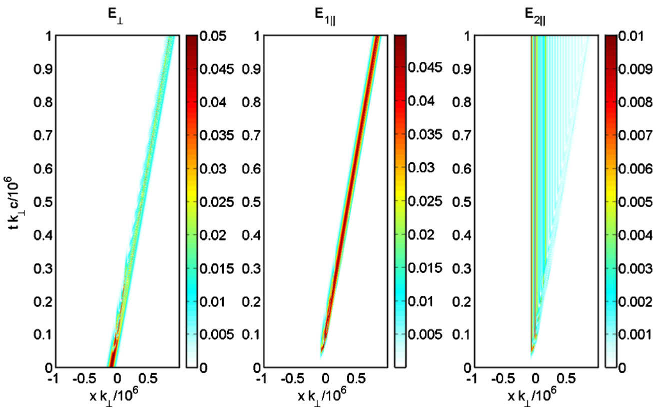

FIG. 3 (color online). The decay of the pump mode $E_\perp = \tilde{E}_\perp / E_{\mathbf{crit}}$ in a strongly magnetized plasma. Initially the localized pump was set to $E_\perp = 0.05 \exp[-(xk_\perp - 10^5)^2/2.5 \times 10^9]$. The pump decays into a forward scattered wave $E_{1\parallel}$ and a mode $E_{2\parallel}$, which has almost zero group speed. We used $\xi = 0.01$ and $\omega_p = 3\xi\omega_\perp = 0.03\omega_\perp$, yielding $k_{1\parallel} = k_\perp$, $k_{2\parallel} \approx 0$, $\omega_\perp = 0.959 k_\perp c$, $\omega_{1\parallel} = 0.929 k_\perp c$, and $\omega_{2\parallel} = \omega_p \approx 0.030 k_\perp c$.

increases, plasma effects become important, as depicted in Fig. 3. For $\omega_p \sim \xi\omega_{\perp\text{char}}$, there are two simultaneous effects of the plasma. First, in this regime the QED and plasma contributions to the coupling strength are comparable, i.e., $C_{\text{pl}} \sim C_{\text{QED}}$, which increases the total coupling strength $C = C_{\text{pl}} + C_{\text{QED}}$, since the phases of C_{pl} and C_{QED} coincide. Second, when the group velocity of the backscattered EM wave is slowed down, the effectiveness of convective stabilization is diminished, increasing the interaction strength and speeding up the conversion of the EM wave energy. Similarly to the case without the plasma (Fig. 2), the EM wave energy mainly ends up in the forward scattered EM mode, but the characteristic time scale is considerably faster with the plasma present. We stress that the simulations presented in this Letter contain results that are easily generalizable to other parameter ranges, following the discussion presented above. The combined effect of plasma and QED effects should influence the emission spectra from magnetars and pulsars. While the QED effects alone shift the spectrum towards linear polarization, we emphasize that the effect is much more pronounced when plasma effects are present, which holds for radiation with frequencies of the order $\omega_{\text{char}} \sim \omega_p/\xi$, i.e., in the optical range for magnetars, while in the infrared to microwave range for pulsars. Thus, we suggest that evidence for a combined plasma-QED photon-splitting process should be sought for in the polarization signature of magnetar and pulsar emission.

[1] M. Marklund and P. K. Shukla, Rev. Mod. Phys. **78**, 591 (2006).
[2] V. B. Berestetskii *et al.*, *Quantum Electrodynamics* (Butterworth-Heinemann, Washington, DC, 1982).
[3] F. Moulin and D. Bernard, Opt. Commun. **164**, 137 (1999); G. Brodin *et al.*, Phys. Rev. Lett. **87**, 171801 (2001); B. Shen *et al.*, Phys. Plasmas **10**, 4570 (2003); S. V. Bulanov *et al.*, Phys. Rev. Lett. **91**, 085001 (2003); Lundström *et al.*, Phys. Rev. Lett. **96**, 083602 (2006).
[4] M. Marklund *et al.*, Phys. Rev. Lett. **91**, 163601 (2003).
[5] M. G. Baring and A. K. Harding, Astrophys. J. **547**, 929 (2001).
[6] C. Kouveliotou *et al.*, Nature (London) **393**, 235 (1998).
[7] R. Baier and P. Breitenlohner, Acta Phys. Austriaca **25**, 212 (1967); Nuovo Cimento B **47**, 117 (1967); I. A. Batalin and A. E. Shabad, Zh. Eksp. Teor. Fiz. **60**, 894 (1971) [Sov. Phys. JETP **33**, 483 (1971)]; A. E. Shabad, Lett. Nuovo Cimento **3**, 457 (1972); D. H. Constantinescu, Nucl. Phys. **B44**, 288 (1972); Lett. Nuovo Cimento **5**, 766 (1972); W.-Y. Tsai, Phys. Rev. D **10**, 2699 (1974); A. E. Shabad, Ann. Phys. (N.Y.) **90**, 166 (1975).
[8] Z. Bialynicka-Birula and I. Bialynicki-Birula, Phys. Rev. D **2**, 2341 (1970).
[9] S. L. Adler, Ann. Phys. (N.Y.) **67**, 599 (1971).
[10] M. G. Baring and A. K. Harding, Astrophys. J. **507**, L55 (1998).
[11] P. Bakshi *et al.*, Phys. Rev. D **14**, 2532 (1976); R. A. Cover *et al.*, *ibid.* **20**, 3015 (1979).
[12] H. Péres Rojas and A. E. Shabad, Ann. Phys. (N.Y.) **121**, 432 (1979); **138**, 1 (1982).
[13] A. E. Shabad and V. V. Usov, Nature (London) **295**, 215 (1982).
[14] V. I. Beskin *et al.*, *Physics of the Pulsar Magnetosphere* (Cambridge University Press, Cambridge, UK, 1993).
[15] D. Lai and W. C. G. Ho, Astrophys. J. **566**, 373 (2002); **588**, 962 (2003); A. Y. Potekhin *et al.*, Astrophys. J. **612**, 1034 (2004).
[16] G. Brodin *et al.*, New J. Phys. **8**, 16 (2006).
[17] W. Heisenberg and H. Euler, Z. Phys. **98**, 714 (1936).
[18] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[19] The cold collisionless momentum equation is valid for $\nu_{\text{ep}} \ll \omega_p$, $v_{\text{th}} \ll c$, conditions fulfilled for a broad range of densities and temperatures. Here ν_{ep} is the collision frequency and v_{th} is the thermal velocity.
[20] The dispersion relation is much simpler for perpendicular propagation, but the main features remain for arbitrary propagation directions [cf. (12) and (13) in Ref. [16]]: the combined QED/plasma effect is important for one of the polarizations, while only QED affects the other.
[21] While accurate in the regime $\omega_p \ll \xi\omega_\perp$, the linear QED contribution in principle needs to be modified for the lf-mode when $\omega_p \gtrsim \xi\omega_\perp$. However, the plasma contribution dominates the linear regime in this case, as $\xi \ll 1$, and Eqs. (3) and (4) are thus sufficient also when $\omega \to \omega_p$.
[22] M. Marklund *et al.*, Plasma Phys. Controlled Fusion **47**, L25 (2005).
[23] The plasma nonlinearities are related to a nonlinear conductivity $\boldsymbol{\sigma}_{\text{nl}}$, defined by $j_{\text{nl}\perp} = \sigma_{\text{nl}\perp 12} E_{1\parallel} E_{2\parallel}$ etc., i.e., $\sigma_{\text{nl}\perp 12} = \omega_\perp \epsilon_0 C_{\text{pl}} / E_{\text{crit}}$.
[24] We note that $\lim_{\hbar \to 0}(C_{\text{pl}}/E_{\text{cr}})$ is finite [cf. (4)].
[25] P. Goldreich and W. J. Julian, Astrophys. J. **157**, 869 (1969).
[26] E. Asseo, Plasma Phys. Controlled Fusion **45**, 853 (2003).