An Electroweak Oscillon

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A numerical simulation of the full bosonic sector of the $SU(2) \times U(1)$ electroweak standard model in $3 + 1$ dimensions demonstrates the existence of an oscillon—an extremely long-lived, localized, oscillatory solution to the equations of motion—when the Higgs mass is equal to twice the $W^{\pm}$ boson mass. The oscillon contains total energy 7 TeV localized in a region of radius 0.05 fm.

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Introduction.—In nonlinear field theories, static solitons have been well studied (see, for example, [1]). However, a much broader class of theories contain oscillons, solutions to the equations of motion that are localized in space but oscillate in time. In some special cases, such as the sine-Gordon breather [2] and Q ball [3], conserved charges guarantee the existence of exact, periodic solutions. Even in the absence of such guarantees, however, localized solutions have been found in many theories that either live indefinitely or for extremely long times compared to the natural time scales of the system.

For scalar theories in one space dimension, oscillons have been found to remain periodic to all orders in a perturbative expansion [2] and are never seen to decay in numerical simulations [4], but can decay after extremely long times via nonperturbative effects [5] or by coupling to an expanding background [6]. In both ϕ^4 theory in two dimensions [7,8] and the Abelian Higgs model in one dimension [9], there exist oscillons that are not observed to decay. In ϕ^4 theory in three dimensions [10], there exist long-lived quasiperiodic solutions whose lifetime depends sensitively on the initial conditions. Similar behavior is present in other scalar theories in three dimensions [11] and in higher dimensions [12]. Phenomenologically, small Q balls were considered as dark matter candidates in [13], axion oscillons were studied in [14], and the role of oscillons in and after inflation was investigated in [15]. Oscillons have also been studied in phase transitions [16], monopole systems [17], QCD [18], and gravity [19].

A recent numerical analysis [20] found oscillons in spontaneously broken $SU(2)$ gauge theory with a fundamental Higgs boson whose mass is exactly twice that of the gauge bosons. Current work [21] is investigating an analytic explanation of this mass relationship using a small amplitude analysis [2,18,22], in which the 2:1 ratio arises as a resonance condition necessary for quadratic nonlinear terms to balance dispersive linear terms in the equations of motion. In this analysis, a field of mass m oscillates with amplitude ϵm, frequency $m\sqrt{1-\epsilon^2}$, and length scale $1/(\epsilon m)$. A similar mass relation arises in the study of embedded defects [23]. The field configurations in [20] were restricted to the spherical ansatz [24], meaning they were invariant under combined rotations in space and isospin. Here we extend this analysis to a fully three-dimensional spatial lattice, eliminating rotational symmetry assumptions, and include the $U(1)$ hypercharge field to obtain the full electroweak sector of the standard model without fermions. We use the same $SU(2)$ coupling g and Higgs self-coupling λ as in the pure $SU(2)$ theory, meaning that the Higgs boson mass is twice the mass of the $W^{\pm}$ bosons, and set the $U(1)$ coupling g' so that the Z^0 boson mass matches its observed value.

While one might expect the oscillon to decay rapidly by emitting electromagnetic radiation, it does not. Instead, after initially shedding some energy into electromagnetic radiation, the system settles into a stable, localized oscillon solution that no longer radiates. Similar behavior was observed both when an additional massless scalar field was coupled to breathers in one-dimensional ϕ^4 theory and when an additional spherically symmetric massless scalar field was coupled to oscillons in the spherical ansatz, results that provided motivation for this work.

Continuum theory.—We consider the standard classical $SU(2) \times U(1)$ electroweak theory (see, for example, [25]), ignoring fermions. The Lagrangian density is $\mathcal{L} = (D_\mu\Phi)^\dagger D^\mu\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} - \lambda(|\Phi|^2 - v^2)^2$, where boldface refers to isovectors. The Higgs field Φ is a Lorentz scalar carrying $U(1)$ hypercharge $1/2$ and transforming under the fundamental representation of $SU(2)$. The $SU(2)$ and $U(1)$ field strengths are $\boldsymbol{F}_{\mu\nu} = \partial_\mu\boldsymbol{W}_\nu - \partial_\nu\boldsymbol{W}_\mu + g\boldsymbol{W}_\mu\times\boldsymbol{W}_\nu$ and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and the covariant derivatives are $D_\mu\Phi = (\partial_\mu + ig'B_\mu/2 + ig\boldsymbol{\tau}\cdot\boldsymbol{W}_\mu/2)\Phi$ and $D^\mu\boldsymbol{F}_{\mu\nu} = \partial^\mu\boldsymbol{F}_{\mu\nu} + g\boldsymbol{W}^\mu\times\boldsymbol{F}_{\mu\nu}$, where $\boldsymbol{\tau}$ represents the weak isospin Pauli matrices. The equations of motion are $D_\mu\boldsymbol{F}^{\mu\nu} = \boldsymbol{J}^\nu$, $\partial_\mu F^{\mu\nu} = J^\nu$, and $D^\mu D_\mu\Phi = 2\lambda(v^2 - |\Phi|^2)\Phi$, where $J_\nu = g'\mathrm{Im}(\partial_\nu\Phi)^\dagger\Phi$ and $\boldsymbol{J}_\nu = g\mathrm{Im}(D_\nu\Phi)^\dagger\boldsymbol{\tau}\Phi$.

We work in the gauge $B_0 = 0$, $\boldsymbol{W}_0 = 0$. Then we can apply a Hamiltonian formalism with energy density

$$u = \frac{1}{2}\sum_{j=x,y,z}\left[\dot{B}_j^2 + \dot{\boldsymbol{W}}_j\cdot\dot{\boldsymbol{W}}_j + \sum_{k>j}(F_{kj}^2 + \boldsymbol{F}_{kj}\cdot\boldsymbol{F}_{kj})\right] + |\dot{\Phi}|^2 + \sum_{j=x,y,z}(D_j\Phi)^\dagger(D_j\Phi) + \lambda(|\Phi|^2 - v^2)^2, \quad (1)$$

where the overdot indicates the time derivative. The Gauss's Law constraints, $\sum_j \partial_j \dot{B}_j = J_0$ and $\sum_j D_j \dot{\boldsymbol{W}}_j = \boldsymbol{J}_0$, remain true at all times, at all points in space, assuming they are obeyed by the initial value data.

Lattice theory.—We use the standard Wilsonian approach [26] for implementing gauge fields on the lattice (for a review see [27]), adapted to Minkowski space evolution as in [28]. (The details of the discretization have been modified slightly for the present application.) The $U(1)$ and $SU(2)$ gauge fields live on the links of the lattice and the fundamental Higgs field lives at the lattice sites. The lattice spacing is Δx, and we determine the values of the fields at time $t_+ = t + \Delta t$ based on their values at times t and $t_- = t - \Delta t$. We associate with the link emanating from lattice site p in the positive jth direction the Wilson line $U_j^p = e^{ig'B_j^p\Delta x/2}e^{ig\boldsymbol{W}_j^p\cdot\boldsymbol{\tau}\Delta x/2}$. For the link emanating from lattice site p in the negative jth direction, we take $U_{-j}^p = (U_j^{p-j})^\dagger$, where $p \pm j$ indicates the adjacent lattice site to p, in direction $\pm j$. At the edges of the lattice we use periodic boundary conditions.

The equation of motion for the Higgs field at site p is $\Phi^p(t_+) = 2\Phi^p - \Phi^p(t_-) + \Delta t^2\ddot{\Phi}^p$, where

$$\ddot{\Phi}^p = \sum_{j=\pm x,\pm y,\pm z}\frac{U_j^p\Phi^{p+j} - \Phi^p}{\Delta x^2} + 2\lambda(v^2 - |\Phi^p|^2)\Phi^p, \quad (2)$$

and all fields are evaluated at time t unless otherwise indicated. For the gauge fields, we have

$$U_j^p(t_+) = \exp\left[\ln U_j^p U_j^p(t_-)^\dagger + \frac{\Delta x\Delta t^2}{2i}(J_j^p + \boldsymbol{J}_j^p\cdot\boldsymbol{\tau}) - \frac{\Delta t^2}{\Delta x^2}\sum_{j'\neq j}(\ln U_{\square j,j'}^p + \ln U_{\square j,-j'}^p)\right]U_j^p, \quad (3)$$

where the currents are $J_j^p = g'\mathrm{Im}(\Phi^p)^\dagger U_j^p\Phi^{p+j}/\Delta x$ and $\boldsymbol{J}_j^p = g\mathrm{Im}(\Phi^p)^\dagger\boldsymbol{\tau}U_j^p\Phi^{p+j}/\Delta x$, and we have defined $\ln U_j^p = i\Delta x(g'B_j^p + g\boldsymbol{W}_j^p\cdot\boldsymbol{\tau})/2$ and $U_{\square j,j'}^p = U_j^p U_{j'}^{p+j}U_{-j}^{p+j+j'}U_{-j'}^{p+j'}$. By defining

$$\|U_j^p\|^2 = \frac{|\mathrm{Tr}\ln U_j^p|^2}{g'^2\Delta x^2} + \frac{|\mathrm{Tr}\boldsymbol{\tau}\ln U_j^p|^2}{g^2\Delta x^2} = |B_j^p|^2 + |\boldsymbol{W}_j^p|^2, \quad (4)$$

we can write the energy density as

$$u^p = \frac{1}{2}\sum_{j=x,y,z}\left[\frac{\|\exp(\ln U_j^p(t_+) - \ln U_j^p(t_-))\|^2}{4\Delta t^2} + \sum_{j'>j}\frac{\|U_{\square j,j'}^p\|^2}{\Delta x^2} + \frac{|U_j^p\Phi^{p+j} - \Phi^p|^2}{\Delta x^2}\right] + \frac{|\Phi^p(t_+) - \Phi^p(t_-)|^2}{4\Delta t^2} + \lambda(|\Phi^p|^2 - v^2)^2. \quad (5)$$

At each lattice point, Gauss's Law reads

$$\sum_{j=x,y,z}\frac{\ln U_j^p(t_+)U_j^p(t_-)^\dagger + \ln U_{-j}^p(t_+)U_{-j}^p(t_-)^\dagger}{i\Delta x^2\Delta t} - (J_0^p + \boldsymbol{J}_0^p\cdot\boldsymbol{\tau}) = 0, \quad (6)$$

where $J_0 = g'\mathrm{Im}\{[\Phi^p(t_+) - \Phi^p(t_-)]^\dagger\Phi^p\}/(2\Delta t)$ and $\boldsymbol{J}_0 = g\mathrm{Im}\{[\Phi^p(t_+) - \Phi^p(t_-)]^\dagger\boldsymbol{\tau}\Phi^p\}/(2\Delta t)$.

Numerical simulation.—The initial conditions are obtained starting from an approximate fit to the oscillons found in [20]. That analysis used the spherical ansatz for $SU(2)$ Higgs-gauge theory,

$$\boldsymbol{\tau}\cdot\boldsymbol{W}_i = \frac{1}{g}\left[a_1\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\hat{x}_i + \frac{\alpha}{r}(\tau_i - \boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}\hat{x}_i) - \frac{\gamma}{r}(\hat{\boldsymbol{x}}\times\boldsymbol{\tau})_i\right],$$
$$\Phi = \frac{1}{g}[\mu - i\nu\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}}]\begin{pmatrix}0\\1\end{pmatrix}, \quad (7)$$

where $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$ and a_1, μ, ν, α, and γ are functions of the radius r and time t. The spherical ansatz field definitions have been chosen to match those used in [20], though our conventions for the three-dimensional theory are slightly different. This form provides initial data for the $\boldsymbol{W}$ and Φ fields, with the initial B field chosen to vanish. To guarantee that the initial configuration obeys Gauss's Law in the full $SU(2)\times U(1)$ theory, we generate the fit at a point in the cycle where the time derivatives are smallest, and then set all the initial time derivatives to zero. We work in units where $v = 1/\sqrt{2}$. Since we are dealing with classical dynamics, we can rescale the fields to fix the $SU(2)$ coupling constant at $g = \sqrt{2}$, so that the $W^\pm$ mass is $m_W = gv/\sqrt{2} = 1/\sqrt{2}$. We choose $\lambda = 1$, giving a Higgs boson mass that is twice the $W^\pm$ mass, $m_H = 2v\sqrt{\lambda} = \sqrt{2}$. These choices agree with [20], except v here is $v/\sqrt{2}$ there. Finally, we fix $g' = 0.773$, so that the Z^0 boson has its observed mass.

In these units, we consider initial configurations

$$a_1(r) = \epsilon(0.117\epsilon + 0.016\epsilon r)(\mathrm{sech}2\epsilon r)^{1/8},$$
$$\mu(r) = 1 - 0.138\epsilon\,\mathrm{sech}\frac{\epsilon r}{6.75}, \quad \nu(r) = 0.017\epsilon r\,\mathrm{sech}\frac{\epsilon r}{5},$$
$$\alpha(r) = 0.117\epsilon^2 r\,\mathrm{sech}\frac{\epsilon r}{8}, \quad \gamma(r) = 0, \quad (8)$$

where the adjustable parameter ϵ allows us to include a combined rescaling of the fields' amplitudes and r dependence, as is commonly used in a small amplitude analysis

[2,18,22]. While $\epsilon = 1$ gives an approximation to the spherical ansatz solution of [20], a slightly larger value appears to be necessary for the configuration to settle into a stable solution in the full $SU(2) \times U(1)$ model. The first term in parentheses in the definition of $a_1(r)$ is scaled to match the coefficient of α, ensuring that α, $a_1 - \alpha/r$, γ/r, and ν all vanish as $r \to 0$, as required for regularity of the fields. Within the $SU(2)$ spherical ansatz simulation, these initial conditions settle into a long-lived oscillon, which we never see decay. The $U(1)$ interactions break the grand spin symmetry of the spherical ansatz, though the continuum theory still preserves invariance under combined space and isospin rotations around the z axis. The Cartesian lattice provides a small breaking of all rotational symmetries. Thus configurations at later times are not constrained to lie within this reduced ansatz. [The full three-dimensional simulation does continue to agree with the spherical ansatz simulation when the $U(1)$ interaction is turned off.]

We start from these initial conditions and let the system evolve for as long as is practical numerically. One concern is that the outgoing radiation emitted as the configuration settles into the oscillon solution can wrap around the periodic boundary conditions, return to the region of the oscillon, and potentially destabilize it. However, as long as the region in which the oscillon is localized does not represent a significant fraction of the lattice volume, this radiation is sufficiently diffuse that it does not affect the oscillon. We use a lattice of size $L = 144$ on a side in natural units, which is more than enough to satisfy this criterion. For $L \gtrsim 100$, changing the lattice size simply changes the pattern of noise caused by electromagnetic radiation superimposed on the oscillon region, but does not affect oscillon properties or stability. We can therefore be certain that there is no coherent structure to this unphysical radiation that could possibly be necessary for the oscillon's stability; its only possible effect is to destabilize the oscillon, which only occurs if the radiation is artificially concentrated by a small lattice (e.g., of size $L < 100$). In numerical experiments, these destabilization effects are actually much weaker in the electroweak model than in pure scalar or $SU(2)$ Higgs-gauge models, because in the electroweak model the radiated energy is almost entirely in the electromagnetic field, while the oscillon solution arranges itself to be electrically neutral. For this reason, it is not necessary to use absorptive techniques such as adiabatic damping [7] (which would have to be adapted to accommodate gauge invariance) or an expanding background [6].

We use lattice spacing $\Delta x = 0.75$, though $\Delta x = 0.625$ and $\Delta x = 0.25$ were verified to give completely equivalent results in smaller tests. The time step is $\Delta t = 0.1$. Total energy is conserved to a few parts in 10^3, which is appropriate since our algorithm is second-order accurate. To check Gauss's Law, we square the left-hand side of Eq. (6), take the trace, and then take the square root of the result. The integral of this quantity over the lattice never exceeds 0.025 and shows no upward trend over time, a highly nontrivial check on the numerical calculation. It is necessary, however, to use double precision to avoid very gradual degradation in this result. A run to time 10 000 takes roughly 40 hours using 24 parallel processes, each running on a 2 GHz Opteron processor core.

We follow the energy in a spherical box of radius 28 as the fields are evolved from the initial conditions in Eq. (8). When the Higgs boson mass is twice the $W^\pm$ mass, only a small amount of energy is emitted from the central region, with the rest remaining localized for the length of the simulation. If the masses are not in this ratio, however, the initial configuration quickly disperses. The box radius has been chosen to be just large enough to enclose essentially all of the energy density associated with the stable oscillon solution. As a result, as the initial conditions settle into the stable oscillon solution, we are also able to see a transient "beat" pattern: the field configurations gradually expand and contract slightly over many periods, causing a small amount of energy to move in and out of the box, accompanied by a corresponding modulation of the field amplitudes. (When a larger box size is used, the graph of the energy in the box flattens out.) Similar beats appear in the $SU(2)$ spherical ansatz oscillon [20], but in the electroweak oscillon their amplitude decays more rapidly. Here, as in [20], each excited field oscillates at a frequency just below its mass, with amplitude of order 0.1 and typical radius of order 10. By comparing the total number of cycles to the total time, we find $\omega_H = 1.404$ for the Higgs field components and $\omega_W = 0.702$ for the gauge field components. The primary excitations are in the $W^\pm$ fields and the Φ field, with some energy radiated outward in the electromagnetic field in a dipole pattern and the Z^0 field largely absent. (In the spherical ansatz oscillon the $W^\pm$ and Z^0 fields must appear symmetrically.) The electroweak oscillon remains approximately axially symmetric under combined space and isospin rotations around the z axis. These results suggest a simple modification of the initial configurations in which the τ_z component of $\boldsymbol{W}$ is set to zero in Eq. (7). Then the final oscillon configuration is equivalent, with the same field amplitudes, localized energy and field frequencies, but less energy is shed initially, so there is less superimposed noise caused by radiation returning from the boundaries and the beat pattern is more clearly visible. This case is shown in Fig. 1.

Conclusions.—We have seen strong evidence for the existence of a long-lived, localized, oscillatory solution to the field equations of the bosonic electroweak sector of the standard model in the case where the Higgs boson mass is twice the $W^\pm$ mass. Compared to the natural scales of the system, this solution has fairly small field amplitudes, but because of its large spatial extent it is very massive. Such large, coherent objects are well described by the classical analysis undertaken here. Quantization of the small oscillations around the oscillon solution would nonetheless be of interest, perhaps using methods similar to those applied to Q-ball oscillons in [29].

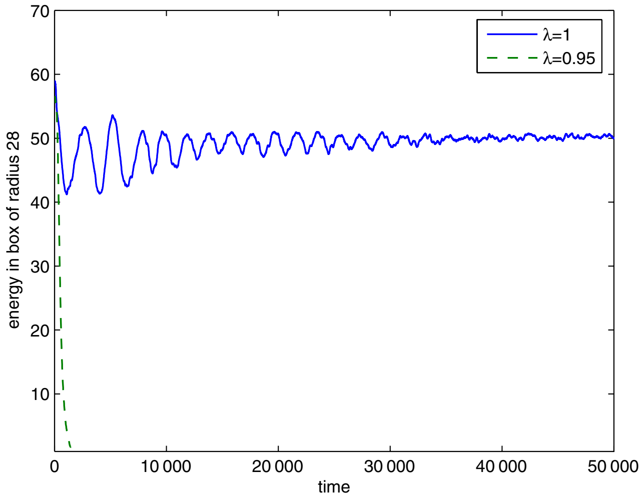

FIG. 1 (color online). Energy in a box of radius 28 as a function of time in the natural units of [20]. One unit of energy is 144 GeV, one unit of time is 5.79×10^{-27} sec, and one unit of length is 1.74×10^{-18} m. The initial conditions are given by Eq. (8) with $\epsilon = 1.15$ and the τ_z component of $\boldsymbol{W}$ in Eq. (7) set to zero. For $\lambda = 1$, the masses of the Higgs and W fields are in a 2:1 ratio and the solution remains localized throughout the simulation. A transient beat pattern is also visible. For $\lambda = 0.95$, the mass ratio is 1.95:1, there is no stable object, and the solution quickly disperses.

Forming oscillons would likely require large energies available only in the early universe. In this context, it would be very desirable to incorporate fermion couplings, which have been ignored here. (Lattice chiral fermions introduce significant, but not insurmountable, technical complications.) While one might expect the oscillon to be destabilized by decay to light fermions, in the case of the photon coupling we have seen that the analogous decay mechanism is suppressed. A slow fermion decay mode would be of particular interest in baryogenesis, since it could provide a mechanism for fermions to be produced out of equilibrium, as is necessary to avoid washout of particle-antiparticle asymmetry. Or, if the oscillon is extremely long-lived, it could provide a dark matter candidate. If such results proved compelling, this analysis would suggest a preferred value of the Higgs boson mass.

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[1] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, England, 1985); R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[2] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **11**, 3424 (1975).
[3] S. Coleman, Nucl. Phys. **B262**, 263 (1985).
[4] D. K. Campbell, J. F. Schonfeld, and C. A. Wingate, Physica (Amsterdam) **D9**, 1 (1983).
[5] H. Segur and M. D. Kruskal, Phys. Rev. Lett. **58**, 747 (1987).
[6] N. Graham and N. Stamatopoulos, Phys. Lett. B **639**, 541 (2006).
[7] M. Gleiser and A. Sornborger, Phys. Rev. E **62**, 1368 (2000).
[8] M. Hindmarsh and P. Salmi, hep-th/0606016.
[9] P. Arnold and L. McLerran, Phys. Rev. D **37**, 1020 (1988).
[10] I. L. Bogolyubsky and V. G. Makhankov, JETP Lett. **24**, 12 (1976); M. Gleiser, Phys. Rev. D **49**, 2978 (1994); E. J. Copeland, M. Gleiser, and H. R. Muller, Phys. Rev. D **52**, 1920 (1995); A. B. Adib, M. Gleiser, and C. A. S. Almeida, Phys. Rev. D **66**, 085011 (2002); E. P. Honda and M. W. Choptuik, Phys. Rev. D **65**, 084037 (2002); G. Fodor *et al.*, hep-th/0609023.
[11] B. Piette and W. J. Zakrzewski, Nonlinearity **11**, 1103 (1998).
[12] M. Gleiser, Phys. Lett. B **600**, 126 (2004); P. M. Saffin and A. Tranberg, hep-th/0610191.
[13] A. Kusenko and M. E. Shaposhnikov, Phys. Lett. B **418**, 46 (1998); K. Enqvist and J. McDonald, Phys. Lett. B **425**, 309 (1998); S. Kasuya and M. Kawasaki, Phys. Rev. D **61**, 041301 (2000).
[14] E. W. Kolb and I. I. Tkachev, Phys. Rev. D **49**, 5040 (1994).
[15] M. Broadhead and J. McDonald, Phys. Rev. D **72**, 043519 (2005); M. Gleiser, hep-th/0602187.
[16] M. Gleiser and R. C. Howell, Phys. Rev. E **68**, 065203(R) (2003); Phys. Rev. Lett. **94**, 151601 (2005).
[17] G. Fodor and I. Rácz, Phys. Rev. Lett. **92**, 151801 (2004); P. Forgács and M. S. Volkov, Phys. Rev. Lett. **92**, 151802 (2004); G. Fodor and I. Rácz, hep-th/0609110.
[18] J. N. Hormuzdiar and S. D. H. Hsu, Phys. Rev. C **59**, 889 (1999).
[19] I. Dymnikova *et al.*, Grav. Cosmol. **6**, 311 (2000).
[20] E. Farhi *et al.*, Phys. Rev. D **72**, 101701(R) (2005).
[21] E. Farhi, N. Graham, A. H. Guth, R. R. Rosales, and R. Stowell (to be published).
[22] A. M. Kosevich and A. S. Kovalev, Sov. Phys. JETP **40**, 891 (1975); R. R. Rosales (private communication).
[23] N. F. Lepora, Phys. Lett. B **541**, 362 (2002).
[24] R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D **10**, 4138 (1974); B. Ratra and L. G. Yaffe, Phys. Lett. B **205**, 57 (1988).
[25] F. Mandl and G. Shaw, *Quantum Field Theory* (John Wiley & Sons, New York, 1984).
[26] K. Wilson, Phys. Rev. D **10**, 2445 (1974).
[27] J. Smit, *Introduction to Quantum Fields on a Lattice* (Cambridge University Press, Cambridge, England, 2001).
[28] J. Ambjorn *et al.*, Nucl. Phys. **B353**, 346 (1991); A. Tranberg and J. Smit, J. High Energy Phys. 11 (2003) 016.
[29] N. Graham, Phys. Lett. B **513**, 112 (2001).