Keeping a Quantum Bit Alive by Optimized π-Pulse Sequences

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A general strategy to maintain the coherence of a quantum bit is proposed. The analytical result is derived rigorously including all memory and backaction effects. It is based on an optimized π-pulse sequence for dynamic decoupling extending the Carr-Purcell-Meiboom-Gill cycle. The optimized sequence is very efficient, in particular, for strong couplings to the environment.

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Quantum information processing is a very promising and very challenging concept [1]. The basic feature which makes quantum information conceptually more powerful than classical information is the quantum mechanical superposition principle. It allows for the parallel processing of many classical registers—the so-called quantum parallelism. The single quantum bit (qubit) is a two-level system which we may identify with a $S = 1/2$ system with states $\downarrow$ and $\uparrow$. Henceforth, we will use this spin language to characterize the qubit. In order for this idea to work, the qubit has to maintain its quantum state not only with respect to the state $\uparrow$ or $\downarrow$ but also with respect to its relative phase. Unavoidable couplings between the qubit and the environment spoil the quantum state: the qubit loses its coherence. This decoherence is one of the most serious obstacles on the way towards applications. Hence finding strategies to suppress decoherence is a crucial field of research [1].

Dynamic decoupling [2–5] is one means to fight decoherence. The idea comes from spin-echo pulses in NMR [6] where a large ensemble of spins is considered. Static but nonuniform couplings can be compensated perfectly by a single π pulse in the middle of the elapsing time interval. The detrimental effect of more complicated perturbations like dynamic interactions with the environment can be suppressed by periodic π pulses or by periodic Carr-Purcell cycles of two π pulses each [6].

The aim of this Letter is to show that an optimized sequence of π pulses suppresses the decoherence even more efficiently than the so far known sequences of equidistant pulses [2–5]. The proposed scheme extends the known Carr-Purcell-Meiboom-Gill (CPMG) cycle [6,7]. In particular, for a strong coupling to the environment, the optimized sequence achieves a much better suppression for the same number of pulses. So the optimized sequences will help to come closer to the realization of quantum information devices.

We consider a fully quantum mechanical model

$$H = \sum_i \omega_i b_i^\dagger b_i + \frac{1}{2}\sigma_z \sum_i \lambda_i (b_i^\dagger + b_i) + E \tag{1}$$

consisting of a single qubit interpreted as spin $S = 1/2$, whose operators are the Pauli matrices σ_x, σ_y, and σ_z. The environment is represented by a bosonic bath with annihilation (creation) operators $b_i^{(\dagger)}$. The constant E sets the energy offset. The relevant bath properties are given by the spectral density [8,9]

$$J(\omega) = \sum_i |\lambda_i|^2 \delta(\omega - \omega_i) \tag{2a}$$

$$= 2\alpha\omega\Theta(\omega_D - \omega), \tag{2b}$$

where we have chosen the standard Ohmic bath with linearly rising density in (2b); α is the dimensionless parameter controlling the coupling between qubit and bath. But our scheme proposed below can be applied to any spectral density $J(\omega)$. The high-energy cutoff ω_D is chosen as in a Debye model for phonons, but other choices are equally possible. Note that the correlation time t_C of the bath is set by $1/\omega_D$.

The model (1) can be easily diagonalized by the unitary transformation $U = \exp(\sigma_z K)$, where $K = \sum_i [\lambda_i/(2\omega_i)] \times (b_i^\dagger - b_i)$ is anti-Hermitean. The resulting effective Hamiltonian [10] $H^{\rm ef} = \sum_i \omega_i b_i^\dagger b_i + \Delta E$ is manifestly diagonal with $\Delta E = E - \int_0^\infty J(\omega)/\omega d\omega$. In spite of this simplicity, (1) suffices to study decoherence of the T_2-type in the NMR language which corresponds to phase decoherence in the XY plane of the impurity spin. In this sense, (1) constitutes a minimal, but fully quantum mechanical, model to investigate decoherence phenomena. Spin flips, however, do not occur so that T_1 is infinite. The minimal model renders the analytic examination of various pulse sequences possible. All thermal, quantum, or memory effects in the bath as well as backactions of the qubit on the bath are included. The pulses used in the following will always be considered to be ideal, i.e., instantaneous (cf. Refs. [2]) and without any error.

First, we look at a simple measurement assuming initially $\sigma_z = 1$ and the bath to be in its thermal equilibrium. Such a state is generated by applying a sufficiently strong magnetic field in z direction. Then a rotation about σ_x is applied $D_x(\gamma) = \exp(-i\gamma\sigma_x/2) = \cos(\gamma/2) + i\sigma_x \times \sin(\gamma/2)$ which transforms the spin in z direction to

$$D_x(\gamma)^\dagger \sigma_z D_x(\gamma) = \cos\gamma\sigma_z + \sin\gamma\sigma_y. \tag{3}$$

For $\gamma = \pi/2$ a rotation by 90° is achieved; for $\gamma = \pi$ the inversion $\sigma^z \to -\sigma^z$. Measuring σ_y leads to the signal

$$s(t) = \langle\uparrow|D_x(\pi/2)^\dagger \exp(iHt)\sigma_y \exp(-iHt)D_x(\pi/2)|\uparrow\rangle = \langle\uparrow|D_x^{\rm ef}(\pi/2)^\dagger \sigma_y^{\rm ef}(t)D_x^{\rm ef}(\pi/2)|\uparrow\rangle. \quad (4)$$

The brackets stand for the thermal expectation value of the bosonic bath. To obtain the line (4) we transform by U to the effective variables and define $A(t) := \exp(iHt)A\exp(-iHt)$. The spin content of the resulting expression can be calculated using

$$\sigma_x^{\rm ef}(t)|\uparrow/\downarrow\rangle = \exp[\mp 2K(t)]|\downarrow/\uparrow\rangle \quad (5a)$$

$$\sigma_y^{\rm ef}(t)|\uparrow/\downarrow\rangle = \pm i\exp[\mp 2K(t)]|\downarrow/\uparrow\rangle. \quad (5b)$$

The bosonic expectation values of exponentials are computed for operators A, B linear in the bosonic variables with the help of $\exp(A)\exp(B) = \exp(A+B)\exp([A,B]/2)$ and with the help of $\langle\exp(A)\rangle = \exp(\langle A^2\rangle/2)$. In this way, we arrive at

$$s(t) = \cos[2\varphi(t)]\exp[-2\chi(t)] \quad (6a)$$

with

$$\varphi(t) = \frac{1}{2}\int_0^\infty J(\omega)\frac{\sin(\omega t)}{\omega^2}d\omega \quad (6b)$$

$$\chi(t) = \int_0^\infty J(\omega)\frac{\sin(\omega t/2)^2}{\omega^2}\coth(\beta\omega/2)d\omega. \quad (6c)$$

Figure 1 illustrates the effect of the coupling strength α and of finite temperature $T = 1/\beta > 0$. Figures 1(a) and 1(b) display the usual long-time decay while Fig. 1(c) focuses on the deviation $1 - s(t)$ from unity. For quantum information processes Fig. 1(c) shows the relevant data since $1 - s(t)$ should be as low as possible. If error correction is to be applied thresholds between 10^{-4} [11,12] and 10^{-2} [13,14] have to be met. Inspecting Fig. 1(c) we conclude that for values of α of about 0.1 the qubit can be stored only for tiny fractions of the correlation time t_C.

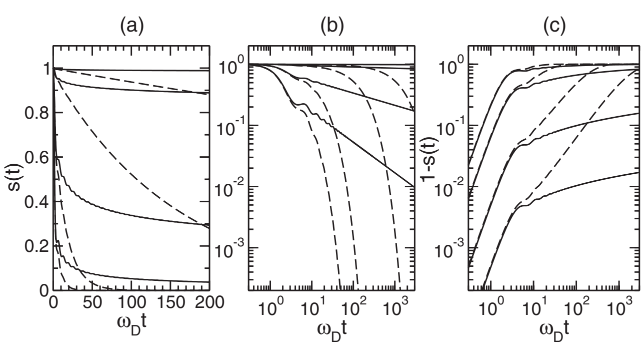

FIG. 1. Signal (4) vs time. Solid lines for $T = 0$; dashed ones for $T = 0.1\omega_D$. Panels (a) (linear) and (b) (double logarithmic) from bottom to top for $\alpha = 0.25$, 0.1, 0.01, 0.001. Panel (c) (double logarithmic) displays $1 - s(t)$ for the same values from top to bottom.

But even if α is significantly smaller, no storage is possible for t_C, let alone for any time longer.

Another interesting conclusion is that *low* values of ω_D are favorable since they set a *long*-time scale [15]. This means that an elastically soft environment, for instance, in an organic compound with low ω_D, is better suited than a hard environment with high ω_D. This is counterintuitive, since one might have suspected that the influence of vibrations is lower when the spring constants $\propto\omega_D^2$ are higher. The objection that the positive effect of a larger t_C in a soft medium will be thwarted by a large value of the coupling α will be invalidated below.

We pass now to a sequence of pulses where the total time interval $0 \to t$ is split into smaller intervals $0 \to \delta_1 t \to \delta_2 t \to \ldots \to \delta_n t \to t$ with $0 < \delta_1 < \delta_2 < \ldots < \delta_n < 1$. The δ values are taken to be fixed. At each instance $\delta_i t$ a π pulse $\sigma_y = -iD_y(\gamma = \pi) = -i\exp(-i\gamma\sigma_y/2)|_{\gamma=\pi}$ is applied which effectively changes the sign of the interaction in Eq. (1). Hence the observable signal changes from $s(t)$ in Eq. (4) to

$$s_n(t) = \langle\uparrow|D_x^{\rm ef}(\pi/2)^\dagger R^\dagger \sigma_y^{\rm ef}(t)RD_x^{\rm ef}(\pi/2)|\uparrow\rangle \quad (7a)$$

$$R = \sigma_y^{\rm ef}(\delta_n t)\sigma_y^{\rm ef}(\delta_{n-1}t)\ldots\sigma_y^{\rm ef}(\delta_2 t)\sigma_y^{\rm ef}(\delta_1 t). \quad (7b)$$

The evaluation of $s_n(t)$ is based on the same identities as the one of $s(t)$ except that it is algebraically more involved. The result is cast in the form

$$s_n(t) = \cos[2\varphi_n(t)]\exp[-2\chi_n(t)] \quad (8a)$$

with

$$\varphi_n(t) = \int_0^\infty \frac{J(\omega)}{2\omega^2}x_n(\omega t)d\omega \quad (8b)$$

$$\chi_n(t) = \int_0^\infty \frac{J(\omega)}{4\omega^2}\coth(\beta\omega/2)|y_n(\omega t)|^2 d\omega, \quad (8c)$$

where the factor $x_n(z)$ in the integrand of the phase reads

$$x_n(z) = (-1)^n\sin(z) + \sum_{m=1}^n (-1)^{m+1}\sin(z\delta_m) \quad (9)$$

and the factor $y_n(z)$ in the integrand of $\chi_n(t)$ reads

$$y_n(z) = 1 + (-1)^{n+1}e^{iz} + 2\sum_{m=1}^n (-1)^m e^{iz\delta_m}. \quad (10)$$

The phase is less harmful since its influence on the signal is only quadratic in α. Hence we focus on χ_n. The procedure discussed so far is based on n equidistant pulses [2,3,5] with $\delta_m = m/(n+1)$ yielding

$$|y_n(z)|^2_{\rm eq} = 4\tan^2[z/(2n+2)]\cos^2(z/2) \quad \forall\ n \text{ even} \quad (11a)$$

$$|y_n(z)|^2_{\rm eq} = 4\tan^2[z/(2n+2)]\sin^2(z/2) \quad \forall\ n \text{ odd}. \quad (11b)$$

For small values of $z/(2n+2)$ these functions rise quadratically like $(1/2)z^2/(n+1)^2$ omitting rapid oscillations. Even without performing the integrations in (8) one can read off two features: (i) a large number n of pulses is advantageous. The time scale t_C is prolonged like $t_C \to (n+1)t_C$. (ii) No further suppression is achieved since the power law $\propto z^2 = (\omega t)^2$ remains unchanged.

Results of the equidistant π-pulse sequence are shown in Fig. 2 as dashed-dotted lines. Clearly, a shift to the right is discernible reflecting the growing factor $n+1$. But no significant change of the slopes occurs. It is still important to have a weak coupling between qubit and bosonic bath to store the qubit for a significant time. For instance, 100 pulses make it possible to store the qubit up to an error of 10^{-4} for $\approx 5t_C$ at $\alpha = 0.25$ while for $\alpha = 0.001$ it may be stored for $\approx 60t_C$.

Now we pose the question whether the sequence of pulses can be optimized, for instance by exploiting the freedom of choosing the instants of the pulses. The potential of nonequidistant pulse sequences was recently demonstrated by concatenated pulse sequences [16]. In our work, we aim at finding the *optimum* pulse sequence with respect to canonical requirements.

Inspecting (10) one realizes that $y_n(0) = 0$ always and that there are n free parameters $\{\delta_j\}$. So one may require that n additional conditions are fulfilled. We use this freedom to make the first n derivatives $y_n^{(j)}(z)|_{z=0}$ with $j \in \{1, 2, \ldots n\}$ vanish. Nicely, the resulting equations have a simple analytic solution for n π pulses

$$\delta_j = \sin^2[\pi j/(2n+2)]. \tag{12}$$

This is the main result of this Letter. The resulting factor in the integrand yields

$$|y_n(z)|^2_{\rm op} = \left| \sum_{j=-n-1}^{n} (-1)^j e^{(iz/2)\cos[\pi j/(n+1)]} \right|^2 \tag{13a}$$

$$\approx 16(n+1)^2 J^2_{n+1}(z/2), \tag{13b}$$

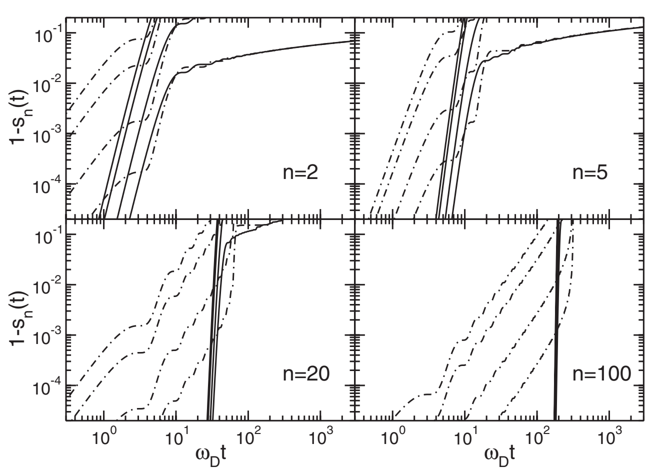

FIG. 2. Signal (8) vs time for various numbers of pulses at $T = 0$. Solid lines for the optimized sequence, dashed-dotted lines for the equidistant sequence (see main text). From top to bottom the curves refer to $\alpha = 0.25$, 0.1, 0.01, 0.001.

where the second line with the Bessel function J_{n+1} represents a very good approximation valid for $z/(2n+2) < 1$ up to exponential corrections. Note that $(n+1)J^2_{n+1}(z/2) \propto [z/(2n+2)]^{2n+2}$ manifesting the effect of the vanishing leading derivatives. From (13) follows that the integrand stays extremely small up to a certain value of z of the order of unity implying that decoherence hardly takes place up to a certain time $t_{\rm op}$ given by $t_{\rm op} \approx (n+1)t_C$. Beyond this time it sets in very abruptly.

How does our finding compare to known results? For $n = 2$ we retrieve from Eq. (12) $\delta_1 = 1/4$ and $\delta_2 = 3/4$. This means that our pulse sequence with the initial $\pi/2$ pulse about σ_x and two π pulses about σ_y reproduces the CPMG cycle [6] which is widely considered for decoherence suppression [7,17]. For all $n > 2$, Eq. (12) predicts so far unexplored pulse sequences with a better potential for decoherence suppression.

Figure 2 depicts the features of the optimum sequence. Clearly, the lines are shifted to the right reflecting the factor $(n+1)$ in $t_{\rm op}$ in parallel to the effect of equidistant pulses. In contrast to equidistant pulses the optimized pulses lead to steeper and steeper slopes on increasing n implying that the behavior for different couplings α becomes almost indistinguishable. This feature is extremely advantageous because it means practically that even a large coupling α does not harm a long storage time. For instance, for 100 pulses the qubit may be stored for $\approx 200t_C$ *independently* of the value of α. The possible storage time is by a factor of 40 better than for the equidistant scheme for $\alpha = 0.25$. For $\alpha = 0.001$ the improvement is still about a factor of 4. Clearly, the improvement is most striking for large values of the coupling. Recurring to the estimate of $t_C = 1$ ps [15] we see that 100 pulses allow us to extend the storage time to about 200 ps.

Another way of looking at the optimized scheme Eq. (12) is to ask how many pulses are needed to achieve a certain storage time with an error below a certain threshold, say 10^{-4}. For $\alpha = 0.25$, 5 or 6 pulses already imply a storage time of $5t_D$. For the same storage, the equidistant scheme requires about 100 pulses. Keeping in mind that in practice each pulse will be imperfect, it is certainly advantageous to work with a minimum number of pulses.

Let us turn to temperature. In practice, no system will be at $T = 0$ and, in particular, the favorable soft media will be operated at relative high T compared to the cutoff temperature $T_D = \omega_D$ (setting $k_B = 1 = \hbar$). In our model, $T = 1/\beta$ enters in the Eqs. (6c) and (8c) by the coth factor reflecting the thermal occupation of the bosonic modes. It deviates from its $T = 0$ value of unity only for small frequencies. Small frequencies mean small values of $z = \omega t$ so that the suppression of $|y_n(z)|^2_{\rm op}$ in this range, see Eq. (13b), is particularly helpful. Finite temperature does not lead to any noticeable decoherence as long as the storage time is not too long, i.e., as long as it stays below $t_{\rm op}$. Indeed, curves at $T \neq 0$ are indistinguishable from the solid ones in Fig. 2. This holds already for a rather small number of pulses so that it is a highly relevant feature for experimental realizations. For the equidistant scheme the thermal effects are larger, in particular, for large temperatures $T \gtrsim T_D$. The extreme insensitivity to thermal effects represents another essential advantage of the optimized scheme.

The above derivations hold for arbitrary T, i.e., even for the classical limit $T \rightarrow \infty$. Indeed, the same pulse sequences can be used for the classical Hamiltonian $H = [f(t)/2]\sigma_z$, where $f(t)$ is controlled by Gaussian fluctuations determined by $\langle f(t)\rangle = 0$ and by $\langle f(t_1)f(t_2)\rangle = g(t_1 - t_2)$. The Fourier transform $p(\omega)$ of $g(t)$ is the power spectrum and $p(\omega)/\pi$ replaces $J(\omega)\coth(\beta\omega/2)$ in Eqs. (6c) and (8c) while the phases φ and φ_n are zero classically. The other equations remain the same, in particular, Eqs. (12) and (13). This observation greatly increases the applicability of our findings since many systems, not only bosonic baths, can be described for high temperatures by classical Gaussian fluctuations.

The equidistant dynamic decoupling or iterated CPMG cycles have been realized experimentally, in particular, in NMR experiments. The detrimental influence of very slow nuclear spins on a solid-state qubit [18] or on the electron spin in quantum dots [19] has been reduced recently. A Rabi oscillation could be made vanish by realizing sequences of almost instantaneous π pulses exploiting the interplay between nuclear and electronic spins [20]. Krojanski and Suter demonstrated recently that even the decoherence of large quantum registers, realized by nuclear spins and their dipole-dipole interaction, can be significantly reduced by dynamic decoupling [21]. To our knowledge, however, no optimized sequences obeying Eq. (12) have been examined.

An optimized sequence is by definition more efficient than a random one; cf. Ref. [22]. But for large symmetry groups it may be easier to use a random scheme than to optimize the pulse sequence. If more specific information on the bath is available, cf. Ref. [17], other, specifically adapted schemes might work more efficiently. Furthermore, we emphasize that hybrid techniques are attractive: any other dynamic decoupling scheme may be improved by replacing the π pulse or the CPMG cycle of two π pulses by a suitable $n > 2$ sequence obeying Eq. (12). The optimized design of real π pulses of finite duration in the presence of bosonic baths, cf. Ref. [23] for classical baths, is left for future research.

In summary, we discussed strategies for suppressing the decoherence of physical quantum bits by dynamic decoupling. A promising way to optimize the sequence of π pulses beyond the well-known CPMG sequence was analytically established. The comparison to equidistant pulse sequences revealed that the optimized scheme enhances the possible storage time by up to almost 2 orders of magnitude. Alternatively, the number of pulses required to achieve a certain prolongation of the storage time can be much smaller (by a factor of 20 for strong coupling to the bosonic bath) than for the standard equidistant scheme. Additionally, the optimized scheme is extremely insensitive to detrimental thermal fluctuations. So experimental investigations of the optimized scheme are called for.

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