Phase Transitions in Systems of Self-Propelled Agents and Related Network Models

M. Aldana,[1,2,*] V. Dossetti,[2] C. Huepe,[3] V. M. Kenkre,[2] and H. Larralde[1]

[1]Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, Cuernavaca, Morelos 62251, México

[2]Consortium of the Americas for Interdisciplinary Science, University of New Mexico, 800 Yale Boulevard NE, Albuquerque, New Mexico 87131, USA

[3]614 N. Paulina Street, Chicago, Illinois 60622-6062, USA

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An important characteristic of flocks of birds, schools of fish, and many similar assemblies of self-propelled particles is the emergence of states of collective order in which the particles move in the same direction. When noise is added into the system, the onset of such collective order occurs through a dynamical phase transition controlled by the noise intensity. While originally thought to be continuous, the phase transition has been claimed to be discontinuous on the basis of recently reported numerical evidence. We address this issue by analyzing two representative network models closely related to systems of self-propelled particles. We present analytical as well as numerical results showing that the nature of the phase transition depends crucially on the way in which noise is introduced into the system.

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The collective motion of a group of autonomous particles is a subject of intense research that has potential applications in biology, physics, and engineering [1–3]. One of the most remarkable characteristics of systems such as a flock of birds, a school of fish, or a swarm of locusts is the emergence of ordered states in which the particles move in the same direction, in spite of the fact that the interactions between the particles are (presumably) of short range. Given that these systems are generally out of equilibrium, the emergence of ordered states cannot be accounted for by the standard theorems in statistical mechanics that explain the existence of ordered states in equilibrium systems typified by ferromagnets.

A particularly simple model to describe the collective motion of a group of self-propelled particles was proposed by Vicsek *et al.* [4]. In this model each particle tends to move in the average direction of motion of its neighbors while being simultaneously subjected to noise. As the amplitude of the noise increases the system undergoes a phase transition from an ordered state in which the particles move collectively in the same direction, to a disordered state in which the particles move independently in random directions. This phase transition was originally thought to be of second order. However, due to a lack of a general formalism to analyze the collective dynamics of the Vicsek model, the nature of the phase transition (i.e., whether it is second or first order) has been brought into question [5].

In this Letter we show that the nature of the phase transition can depend strongly on the way in which the noise is introduced into these systems. We illustrate this by presenting analytical results on two different network systems that are closely related to the self-propelled particle models. We show that in these two network models the phase transition switches from second to first order when the way in which the noise is introduced changes from the one presented in [4] to the one described in [5].

The first network model, which we will refer to as the vectorial network model, consists of a network of N 2D vectors (represented as complex numbers), $\{\sigma_1 = e^{i\theta_1}, \sigma_2 = e^{i\theta_2}, \ldots, \sigma_N = e^{i\theta_N}\}$, all of the same length $|\sigma_n| = v$ and whose angles $\{\theta_1(t), \theta_2(t), \ldots, \theta_N(t)\}$ can change in time. Each vector σ_n interacts with a fixed set of K other vectors, $\{\sigma_{n_1}, \ldots, \sigma_{n_K}\}$, randomly chosen from anywhere in the system. We will call this set of K vectors the inputs of σ_n. Once each vector σ_n has been provided with a fixed set of K input connections, the dynamics of the network are then given by one of the two following interaction rules:

$$\theta_n(t+1) = \mathrm{angle}\left\{\frac{1}{vK}\sum_{j=1}^{K}\sigma_{n_j}(t)\right\} + \eta\xi(t), \tag{1}$$

$$\theta_n(t+1) = \mathrm{angle}\left\{\frac{1}{vK}\sum_{j=1}^{K}\sigma_{n_j}(t) + \eta e^{i\xi(t)}\right\}, \tag{2}$$

where for any vector $\vec{v} = |v|e^{i\phi}$ we define the function $\mathrm{angle}(\vec{v}) = \phi$, and $\xi(t)$ is a random variable uniformly distributed in the interval $[-\pi, \pi]$. The dynamics of the network is fully deterministic for $\eta = 0$ and becomes more random as the parameter η increases. In what follows, we will refer to the quantity $(1/vK)\sum_{j=1}^{K}\sigma_{n_j}$ as the average contribution of the inputs of σ_n.

To quantify the amount of order in the system we define the instantaneous order parameter $\psi(t)$ as

$$\psi(t) = \lim_{N\to\infty}\frac{1}{vN}\left|\sum_{n=1}^{N}\sigma_n(t)\right|. \tag{3}$$

In the limit $t \to \infty$, the instantaneous order parameter $\psi(t)$ reaches a stationary value ψ [4–7]. Thus, in the stationary state all the vectors are aligned if $\psi \sim 1$, whereas if $\psi \sim 0$ the vectors point in random directions.

The interaction rules given in Eqs. (1) and (2) were proposed by Vicsek *et al.* in Ref. [4], and by Grégoire and Chaté in Ref. [5], respectively. The difference between these two interaction rules consists in the way in which the noise is introduced: In Eq. (1) the noise is added outside the angle function; i.e., after the angle function has been applied to the average contribution of the inputs. On the other hand, in Eq. (2) the noise is added inside the angle function; i.e., it is added directly to the average contribution of the inputs. In Ref. [5], Grégoire and Chaté posed the question as to whether these two rules lead to the same type of phase transition.

In this Letter we show that the interaction rules in Eqs. (1) and (2) produce different types of phase transitions in the network systems under consideration, which suggests that a similar effect is being observed in [5] for the self-propelled systems.

Obviously, in the self-propelled particle models the elements do not interact through a network. Instead, they move in a 2D space, each particle interacting locally with the particles that fall within a certain radius. This motion allows particles that are initially far apart to meet, interact, and separate again, giving rise to effective long-range interactions. On the other hand, in our vectorial network model the particles are fixed to the nodes of a network. The long-range correlations produced by the motion of the particles in the self-propelled models are proxied in our network model through randomly choosing the inputs of each element from anywhere in the network. An underlying assumption of our work is that the existence and nature of the phase transition depends mostly on the occurrence of such long-range interactions, and less crucially on whether they are produced by the motion of the particles or by the network topology [6,7]. While the exact relation between these two ways of establishing long-range interactions is not yet known, it has been shown that a strong parallel can be established between them [7,8]. Further, below we show that there are at least two limits in which they are fully equivalent: for large particle speeds and for high densities (see Figs. 1 and 2).

In Ref. [7] it was proven that, as the noise amplitude η increases, the vectorial network model with the interaction rule given as in Eq. (1) undergoes a continuous phase transition from ordered states where $\psi > 0$, to disordered states where $\psi = 0$. Figure 1 shows this phase transition obtained numerically for $N = 20\,000$ and $K = 5$. It also displays the phase transition in the Vicsek model for a system with the same N, a density such that the average number of interactions per particle is also $K = 5$, and increasing particle speeds. As can be seen from Fig. 1, the Vicsek model curves approach continuously the network model curve as $v \to \infty$. This supports the idea that in both cases a second order phase transition is observed when the noise is introduced as in Eq. (1), albeit the finite size effects observed near the critical point.

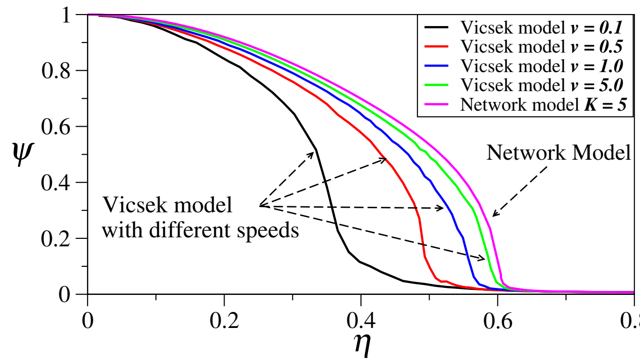

FIG. 1 (color online). Phase diagram of the Vicsek model and the vectorial network model for the case in which the noise is added as in Eq. (1). When the speed v of the particles in the Vicsek model increases, the phase transition converges to that of the vectorial network model. The numerical simulations were carried out for systems with $N = 20\,000$ particles and an average number of interactions per particle $K = 5$.

The probability distribution function (PDF) of the sum $\frac{1}{vK}\sum_{j=1}^{K}\sigma_{n_j}(t) + \eta e^{i\xi(t)}$ that appears in Eq. (2) is computed as for a random walk assuming that all the terms are statistically independent. By projecting this PDF onto the unit circle we can establish a recursion relation for the order parameter, which for $K \gg 1$ becomes $\psi(t+1) = \mathcal{M}_\eta(\psi(t))$, where

$$\mathcal{M}_\eta(\psi(t)) \equiv \begin{cases} \frac{\psi(t)}{2\eta} F_1(\frac{1}{2}, \frac{1}{2}, 2, \frac{[\psi(t)]^2}{\eta^2})2 & \text{if } \psi(t) < \eta \\ 2F_1(\frac{1}{2}, -\frac{1}{2}, 1, \frac{\eta^2}{[\psi(t)]^2}) & \text{if } \eta < \psi(t) \end{cases} \quad (4)$$

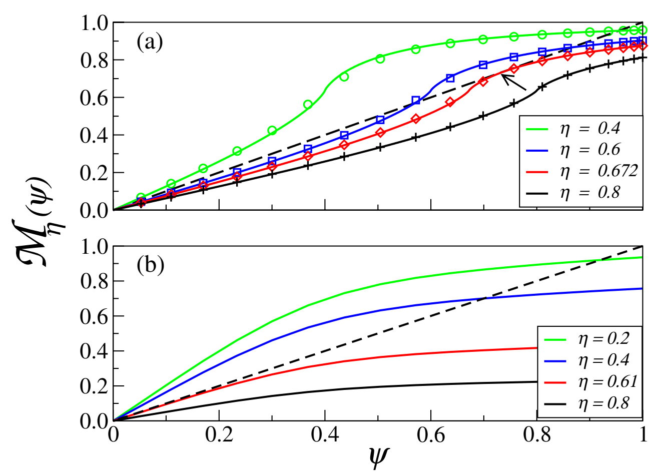

FIG. 2 (color online). Graph of the dynamical mapping $\mathcal{M}_\eta(\psi)$. (a) The interaction rule is as in Eq. (2). The solid curves correspond to the analytical solution given in Eq. (4), and the symbols to the numerical simulation carried out for a system with $N = 20\,000$ and an average number of interactions per particle $K = 100$. (b) The interaction rule is as in Eq. (1). The curves were computed numerically for a system with $N = 20\,000$ particles and $K = 5$. In (a) the nonzero stable fixed point appears discontinuously as η decreases, whereas in (b) it appears continuously.

and ${}_2F_1(a, b; c, x)$ is the Gauss hypergeometric function. The function $\mathcal{M}_\eta(\psi)$ is shown in Fig. 2(a) for different values of η (solid curves). This figure also displays with symbols the numerical dynamical mapping computed for the self-propelled model with the interaction rule given in Eq. (2), $N = 20\,000$ particles, and an average number of interactions per particle $K = 100$. Clearly, the numerical mapping coincides with the theoretical result for $\mathcal{M}_\eta(\psi)$, showing that the network and self-propelled systems are also equivalent in the high density limit case considered here. The numerical mappings for the self-propelled system were obtained by placing the particles in various random initial conditions constrained to produce every order parameter value $\psi(t)$ in the x axis, and then computing one time step using Eq. (2) to obtain the corresponding value $\psi(t+1)$ in the y axis.

The fixed points of the dynamical mapping $\psi(t+1) = \mathcal{M}_\eta(\psi(t))$ give the stationary values of the order parameter. From Eq. (4) it is clear that $\psi = 0$ is always a fixed point. However, the stability of this fixed point changes depending upon the value of η. By numerically solving Eq. (4) to obtain the fixed point, we find that for $0.672 < \eta$ the only stable fixed point is $\psi = 0$. As $\eta \to 0.672$ from above, the graph of $\mathcal{M}_\eta(\psi)$ moves closer to the identity and eventually another nonzero stable fixed point ψ' appears discontinuously when $\eta \approx 0.672$ (see the point indicated with an arrow in Fig. 2(a)]. For $1/2 < \eta \leq 0.672$ there are actually two stable fixed points. In this region of bistability the system shows hysteresis. Finally, when $\eta < 1/2$ the fixed point $\psi = 0$ becomes unstable and only the nonzero fixed point ψ' remains stable. Contrary to this, when the noise is introduced as in Eq. (1) the nonzero stable fixed point appears continuously, as can be seen from Fig. 2(b).

The validity of these results is corroborated by numerical simulations carried out for networks with $N = 10^5$ and $K = 20$. Figure 3 shows the fixed point ψ of Eq. (4) as a function of the noise intensity η (solid line). The discontinuity of the order parameter ψ at $\eta = 0.672$ and $\eta = 1/2$ is apparent. The dashed and dotted-dashed curves are the plots of the results from the numerical simulation for the cases in which all the vectors were initially aligned in the same direction ($\psi(0) = 1$), and when the vectors were initially oriented in random directions ($\psi(0) \simeq 0$), respectively. In the region of bistability $1/2 < \eta < 0.672$, the system reaches one or the other of the two stable fixed points depending upon the initial condition.

The theoretical curves presented in Fig. 3 show the "limits of metastability" for the system, i.e., the maximal and minimal values of η for which the system has bistable behavior (hysteresis). Clearly, specific realizations of the system cannot be driven all the way to the limits of metastability and decay at values of η slightly above $1/2$ and below 0.672, as observed in the graph. Additionally, Eq. (4) was obtained in the limit of large K; however, already for $K = 20$ the agreement is good.

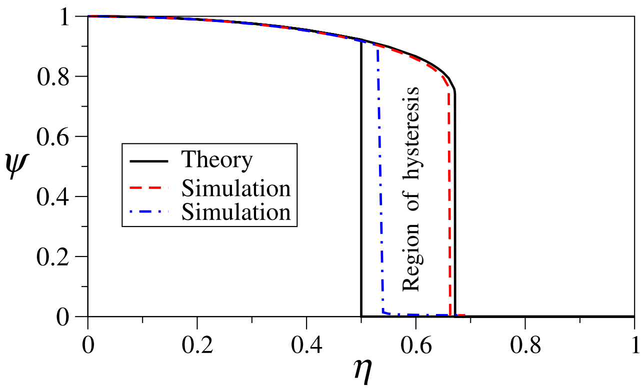

FIG. 3 (color online). Phase diagram of the vectorial network model for the case in which the noise is added as in Eq. (2). The solid line corresponds to the prediction obtained from Eq. (4). The dashed and dotted-dashed curves are the results of the numerical simulation starting out the dynamics from initial conditions for which $\psi(0) \approx 1$ and $\psi(0) \approx 0$, respectively. The phase transition in this case is clearly discontinuous.

The second model that we consider is a majority voter model in which the network elements σ_n can acquire only two values, $+1$ or -1. We can think of this system as a society in which every individual σ_n has to make a decision about an issue with two possible alternatives, either $+1$ or -1. Again, each element σ_n receives inputs from a set of K other elements randomly chosen from anywhere in the system. Let us first consider the case in which the interaction between σ_n and its K inputs, $\{\sigma_{n_1}, \sigma_{n_2}, \ldots, \sigma_{n_K}\}$, is given by

$$\sigma_n(t+1) = \mathrm{sgn}\left[\mathrm{sgn}\left\{\frac{1}{K}\sum_{j=1}^{K}\sigma_{n_j}(t)\right\} + \frac{\xi(t)}{1-\eta}\right], \qquad (5)$$

where $\mathrm{sgn}[x] = 1$ if $x > 0$, $\mathrm{sgn}[x] = -1$ if $x < 0$, η is a parameter that takes a constant value in the interval $[0, 1]$, and $\xi(t)$ is a random variable uniformly distributed between $[-1, 1]$. For the sgn function to be well defined we choose K as an odd integer. Equation (5) is similar to Eq. (1) in that the noise is added to the sign of the average contribution of the inputs. Since in this case σ_n is a discrete variable that takes only the two values $+1$ or -1, the sgn function has to be applied again. This interaction rule reflects the fact that an individual in a society usually tends to be of the same opinion as the majority of his "friends" (inputs), though, with probability $\eta/2$ he can have the opposite opinion.

The instantaneous order parameter $\psi(t)$ is defined as in Eq. (3), but now the vertical bars represent the absolute value instead of the norm of a vector.

In Ref. [9] it has been shown that the majority voter model with the interaction rule given by Eq. (5) undergoes a continuous phase transition as the value of η is varied. In Fig. 4 we reproduce this phase transition for networks with $N = 10^5$ and $K = 3$ (solid curve with circles). It is clear from this figure that, when the noise is added as in Eq. (5),

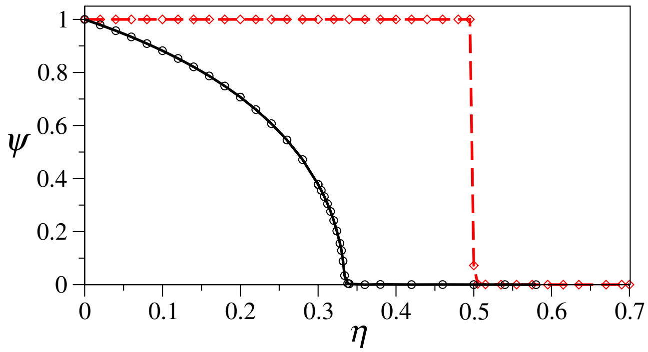

FIG. 4 (color online). Phase transitions in the majority voter model when the noise is added as in Eq. (5) (solid line with circles), and as in Eq. (6) (dashed line with diamonds). The numerical simulations were carried out for systems with $N = 10^5$ and $K = 3$.

the phase transition in the majority voter model is indeed continuous.

Let us now consider the case in which the interaction rule between σ_n and its inputs is given by

$$\sigma_n(t+1) = \mathrm{sgn}\left[\frac{1}{K}\sum_{j=1}^{K}\sigma_{n_j}(t) + 2\eta\xi(t)\right], \qquad (6)$$

where $\xi(t)$ is a random variable uniformly distributed in the interval $[-1, 1]$ and $\eta \in [0, 1]$. This rule is similar to that given in Eq. (2) in that the noise is added to the average contribution of the inputs and then the sgn function is evaluated. Again, the PDF of the sum appearing in Eq. (6) can be computed as for a random walk assuming that all the terms are statistically independent. Integrating the PDF over all positive values of the sum we obtain a recursion relation for the order parameter (see [9]), which for $K = 3$ becomes

$$\psi(t+1) = \begin{cases} \frac{3}{2}\psi(t) - \frac{1}{2}[\psi(t)]^3, & \text{for } 0 < \eta < \frac{1}{6} \\ \frac{1+6\eta}{8\eta}\psi(t) - \frac{1-2\eta}{8\eta}[\psi(t)]^3, & \text{for } \frac{1}{6} < \eta < \frac{1}{2} \\ \frac{1}{2\eta}\psi(t), & \text{for } \frac{1}{2} < \eta \end{cases}$$

Although $\psi = 0$ is always a fixed point of the previous equation, a stability analysis reveals that for $0 < \eta < 1/2$ the solution $\psi = 0$ is unstable and the only stable fixed point is $\psi = 1$. In the region $1/2 < \eta$ the fixed point $\psi = 1$ disappears altogether and the only fixed point that remains is $\psi = 0$, and it is stable. Therefore, at the critical value $\eta = 1/2$ the system undergoes a discontinuous phase transition from a totally ordered state ($\psi = 1$) to a fully random state ($\psi = 0$). Figure 4 shows the results of the numerical simulation for a system with $N = 10^5$ and $K = 3$ (dashed curve with diamonds).

In summary, we have analyzed numerically and analytically the phase transition from ordered to disordered states in two network models that capture some of the main aspects of the interactions in systems of self-propelled particles. In particular, the self-propelled model becomes equivalent to the vectorial network model in the limit of large speeds or high densities. We have shown that for the two network models, the phase transition changes from second order to first order depending on the way in which the noise is introduced into the system. This change is consistent with the results reported by Vicsek *et al.* in [4], and by Grégoire and Chaté in [5]. This consistency suggests that a similar effect is being observed in the self-propelled model and motivates a deeper analysis in order to determine the nature of its phase transition. Clearly, the two ways of introducing noise correspond to different physical situations. On the one hand, with the Vicsek type of noise the uncertainty falls on the decision mechanism. On the other, introducing the noise à la Grégoire-Chaté, the decision function is perfectly determined and the uncertainty falls on the arguments of this function. There is no reason to expect, *a priori*, similar behaviors under these two different physical situations.

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Note added.—While this manuscript was being reviewed, Ref. [10] appeared, in which it is shown that the first-order phase transition found in Ref. [5] by means of the Binder cumulant is a numerical artifact.

*Electronic address: max@fis.unam.mx

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