Universality in a 2-Component Fermi System at Finite Temperature

Gautam Rupak

Institute for Nuclear Theory, University of Washington, Seattle, Washington 98195, USA

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Thermodynamic properties of a Fermi system close to the unitarity limit, where the 2-body scattering length a approaches $\pm\infty$, are studied in the high temperature Boltzmann regime. For dilute systems the virial expansion coefficients in the Boltzmann regime are expected, from general arguments, to be *universal*. A model independent finite temperature T calculation of the third virial coefficient $b_3(T)$ is presented. At the unitarity limit, $b_3^\infty \approx 1.11$ is a universal number. The energy density up to the third virial expansion is derived. These calculations are of interest in dilute neutron matter and could be tested in current atomic experiments on dilute Fermi gases near the Feshbach resonance.

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Recent experiments in atomic traps near the Feshbach resonance [1–5] have opened opportunities to study properties of finite density systems that have not been explored before. Using an external magnetic field, it is possible to tune the 2-body S-wave scattering length a essentially at will: it can be made arbitrarily large ($a = \pm\infty$) compared to the range of the interaction R. Several groups have taken measurements near the Feshbach resonance in ^{6}Li and ^{40}Ka fermionic atomic gases [1–5]. These experiments are important checks of theoretical tools, both new and old, in previously untested territories and hold the prospect of revealing many new phenomenon.

At densities n that are dilute compared to the range of the interactions ($nR^3 \ll 1$), the physics should be insensitive to the details of the interaction at the short distance scale R. Therefore, near the Feshbach resonance, as $|a| \to \infty$, where there are no relevant scales left in the interaction, thermodynamic properties are expected to be *universal* and not just applicable to only atomic systems. Many physical systems in nature are close to this universal limit. In ^{4}He atomic gases, the scattering length $a \sim 100$ Å is much larger than the range of the interaction $R \sim 5$ Å. In nuclear physics, the neutron-neutron S-wave scattering length $a_{nn} \sim -19$ fm is much larger than the range of the interaction set by the pion mass $R \sim \hbar/m_\pi \sim 1.4$ fm. The universal properties learned in atomic experiments are applicable to other problems in the same universality class that are otherwise not directly accessible such as neutrino interactions in dilute neutron matter in supernova [6].

In this Letter, we study the properties of spin-$\frac{1}{2}$ nonrelativistic fermions with large scattering length $|a| \to \infty$ at density $n \ll 1/R^3$ and temperature T such that the thermal wavelength $\lambda = \sqrt{2\pi/(MT)} \gg R$, and we are not sensitive to the short distance scale R. Standard low-density expansion method is not feasible due to the large factors of $|a|^3 n \gg 1$. However, in a window of temperatures T such that $n\lambda^3 \ll 1$ and $\lambda \gg R$, a perturbative calculation in $n\lambda^3 \ll 1$, the Boltzmann regime, is possible even as $|a|^3 n \to \infty$, $|a|/\lambda \to \infty$. The hierarchy of momentum scales is $|a| \gg n^{-1/3} \gg \lambda \gg R$.

In the Boltzmann regime the pressure P of a 2-component Fermi gas can be written in terms of the so-called *virial* expansion [7]:

$$\frac{P}{T} = \frac{2}{\lambda^3}[b_1 z + b_2 z^2 + b_3 z^3 + \ldots], \tag{1}$$

where $z = \exp(\mu/T)$ is the fugacity for the system with chemical potential μ. In this expansion, all the density dependence is in the fugacity z. This is a valid expansion for $z \ll 1$ or equivalently for $n\lambda^3 \ll 1$ as shown later in Eq. (10).

The dimensionless virial coefficients b_n depend on the vacuum interaction and the thermal momentum $\sqrt{MT}$. For a dilute system with thermal wavelength $\lambda \gg R$, the interaction primarily depends on the scattering length a. Thus, as $a \to \pm\infty$ at the unitarity limit, there are no relevant scales left in the interaction and the dimensionless coefficients b_n must be *universal*. This is shown explicitly with a calculation up to the third order in the virial expansion. Small corrections to these universal results from effective range, higher-partial waves, etc., are neglected.

The virial coefficient b_n receives contribution from up to and including n-body physics [7,8]. b_2 is the first coefficient that receives contribution from the interacting theory and it is related to the 2-body scattering phase shift [7–9]:

$$b_2^{(2)} = \frac{1}{\sqrt{2}} e^{\gamma^2/MT}\left[1 + \mathrm{Erf}\left(\frac{\gamma}{\sqrt{MT}}\right)\right], \tag{2}$$

where $\mathrm{Erf}(x)$ is the error function and $\gamma = 1/a$. We use the notation $b_n = b_n^{(1)} + b_n^{(2)} + \ldots + b_n^{(n)}$, where $b_n^{(l)}$ is the nth virial coefficient from l-body interaction. The thermodynamic pressure $P_k^{(l)}$, number density $n_k^{(l)}$, etc., are also denoted in a similar manner. $P^{(l)}$ and $n^{(l)}$ without a subscript denote the total contribution to the pressure and number density from l-body physics. As $|a| \to \infty$, $b_2^{(2)} = 1/\sqrt{2}$ is universal. $b_2^{(2)}$ and $T\partial_T b_2^{(2)}$ are plotted in Fig. 1 near the unitarity limit using the atomic lithium ^{6}Li mass M, at temperatures around 6 μK.

Traditionally the higher virial coefficients are calculated in a cluster expansion with the ''binary collision'' method

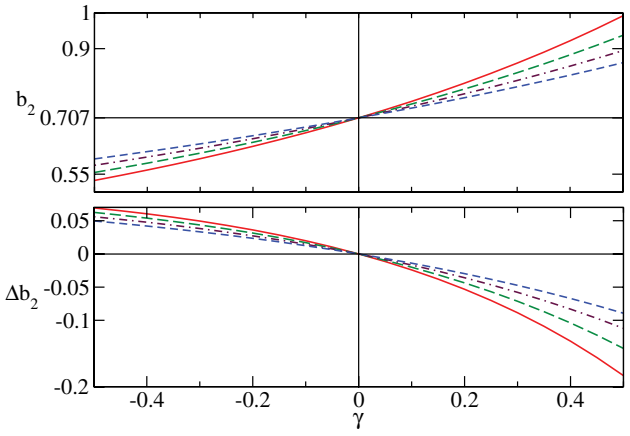

FIG. 1 (color online). Plot of $b_2^{(2)}$ and its derivative $\Delta b_2 \equiv T\partial_T b_2^{(2)}$ for ^{6}Li as a function of the inverse scattering length $\gamma = 1/a$ in eV. 1 eV $\approx 5.06 \times 10^{-4}$ Å^{-1} in natural units ($\hbar = 1 = c$). Solid, long-dashed, dot-dashed, and short-dashed curves correspond to temperatures of 6 μK, 8.4 μK, 11.76 μK, and 16.46 μK, respectively.

of Lee and Yang (see Ref. [10]) but it is not applicable near the unitarity limit with a real or virtual shallow bound state. The third virial coefficient has been calculated for bosons only in very limiting cases [10]. However, it is not clear how the contribution from 3-body bound states associated with the Efimov effect [11] in bosons such as ^{4}He near $|a| \to \infty$ is taken into account. In contrast, the effective field theory calculation takes the ^{4}He 3-body bound state contribution into account directly [8].

The nonrelativistic system near the Feshbach resonance in the S-wave can be described by the Lagrangian density:

$$\mathcal{L} = \psi^\dagger\left(i\partial_0 + \frac{\nabla^2}{2M} + \mu\right)\psi - \frac{g}{4}(\psi\sigma_2\psi)^\dagger(\psi\sigma_2\psi) + \ldots, \tag{3}$$

where the "..." represents higher dimensional operators that are suppressed for dilute systems, ψ are the spin-$\frac{1}{2}$ fermion fields, and the σ_i matrices act in the spin space. The four-fermion coupling g is related to the 2-body scattering length a (see Refs. [12,13] and references therein): $g(\nu) = -4\pi/[M(\nu - 1/a)]$, where ν is the renormalization scale in dimensional regularization.

For bosons, there is a 3-body interaction at leading order [13,14] related to the 3-body bound state that dominates the third virial coefficient b_3 [8]. At the same time this prevents b_3 from being universal for bosons since the 3-body binding energy is system dependent and does not necessarily approach a universal value as $a \to \pm\infty$. The differences between ideal quantum Fermi Eq. (4) and Bose [15] gases become even more significant in the interacting systems.

The first two virial coefficients b_1 and b_2 are known. Nevertheless, it is instructive to derive them in the effective field theory before calculating the third virial coefficient b_3. Feynman diagrams with a closed particle loop are order z (from the energy integral over the Matsubara frequency) and vanish in vacuum as expected. This can be seen by calculating the pressure in the free theory, Fig. 2:

$$P^{(1)} = 2T\sum_{l=-\infty}^{\infty}\int\frac{d^3q}{(2\pi)^3}\log\left[i(2l+1)\pi T + \mu - \frac{\boldsymbol{q}^2}{2M}\right]$$
$$= \frac{2T}{\lambda^3}\left[z - \frac{z^2}{4\sqrt{2}} + \frac{z^3}{9\sqrt{3}} - + \ldots\right], \tag{4}$$

resulting in an ideal Fermi gas "effective" pressure $P/n = T[1 + n\lambda^3/(8\sqrt{2}) + \cdots]$ which is larger than the classical ideal gas [15] due to Pauli's exclusion principle. The classical ideal gas law $P/n = T$ is recovered as $T \to \infty$. A closed particle-particle diagram (only possible with interactions) is $\mathcal{O}(1)$ and does not vanish in the vacuum. However, a closed loop with "baryon number" 2, i.e., a closed dimer propagator, is $\mathcal{O}(z^2)$. Similarly, a closed trimer propagator is $\mathcal{O}(z^3)$. This is more easily demonstrated with the calculation of b_2.

The 2-body scattering is nonperturbative for $|a| \gtrsim \lambda$, and the dimer propagator is given by an infinite sum, Fig. 2:

$$\mathcal{D}(p_0, \vec{\boldsymbol{p}}) \approx \frac{4\pi}{M}\frac{1}{-\frac{1}{a} + \sqrt{\frac{p^2}{4} - Mp_0 - 2M\mu}} + \mathcal{O}(z^2), \tag{5}$$

where $p = |\vec{\boldsymbol{p}}|$, $q = |\vec{\boldsymbol{q}}|$. p_0 is an odd multiple of $i\pi T$ and it is identified with a "final" loop integral variable in Eqs. (6) and (8). The contribution from the dimer propagator to the pressure is

$$P^{(2)} = -T\sum_{l=-\infty}^{\infty}\int\frac{d^3q}{(2\pi)^3}\log\{\mathcal{D}[i(2l+1)\pi T, \vec{\boldsymbol{q}}]\}$$
$$= \frac{1}{2\pi i}\oint\frac{d\eta}{\exp(\eta/T)+1}\int\frac{d^3q}{(2\pi)^3}\log[\mathcal{D}(\eta, \vec{\boldsymbol{q}})]. \tag{6}$$

The contour integral in η is in an anticlockwise sense over

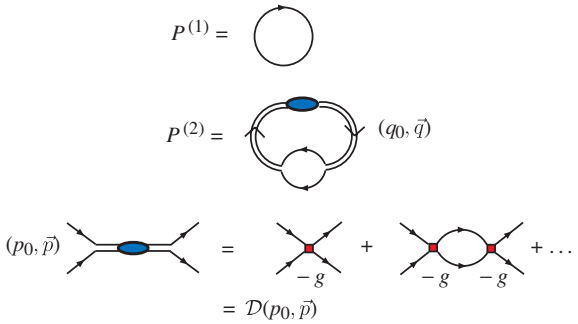

FIG. 2 (color online). The pressure P from 1- and 2-body physics. Solid lines are fermion fields. Each particle-particle loop is order $g\sqrt{MT} \sim a\sqrt{MT}$ and summed to all orders to form the dressed dimer propagator, represented by the double lines.

odd integer multiples of $i\pi T$. Looking at Eqs. (5) and (6), one can see that the energy integral over η is $\mathcal{O}(z^2)$ because a dimer carries chemical potential 2μ. This is similar to the energy η integral for a closed single particle propagator that carries chemical potential μ and $1/[\exp(\eta/T)+1] \sim \mathcal{O}(z)$ for $\eta \sim -\mu$. An explicit calculation shows:

$$P^{(2)} \approx \frac{\sqrt{2}T}{\lambda^3} z^2 e^{\gamma^2/MT}\left[1 + \mathrm{Erf}\left(\frac{\gamma}{\sqrt{MT}}\right)\right] + \mathcal{O}(z^4). \quad (7)$$

Comparing Eqs. (1), (2), (4), and (7), we see that the known results up to the second order in the virial expansion are exactly reproduced in the effective field theory calculation.

The calculation of the third virial coefficient $b_3^{(3)}$ is similar. First, the leading order trimer propagator in the fugacity z expansion is calculated, and then the energy integral over imaginary odd multiples of $i\pi T$ is carried out. The trimer propagator is $\mathcal{O}(1)$ in the z expansion and carries chemical potential 3μ. Thus an energy integration over a closed trimer loop is $\mathcal{O}(z^3)$. The trimer propagator requires summation of an infinite series of Feynman diagrams for $|a|\sqrt{MT/(2\pi)} \gtrsim 1$ that do not reduce to a geometric sum unlike the dimer propagator [13,14]. The relevant Feynman diagrams are shown in Fig. 3. We get:

$$P_3^{(3)} = i\frac{6\sqrt{3}}{\pi^2\lambda^3} z^3 \oint d\eta \int_0^\infty dk \frac{k^2 e^{-\eta/T} a(\eta, k, k)}{k^2 - \frac{4}{3}(M\eta + \gamma^2)}, \quad (8)$$

where the contribution from the trimer propagator, after projecting onto the S wave, is defined through the integral equations

$$a(\eta, k, p) = K(\eta, k, p) + \frac{2}{\pi}\int_0^\infty dl \frac{l^2 a(\eta, k, l) K(\eta, l, p)}{l^2 - \frac{4}{3}(M\eta + \gamma^2)},$$

$$K(\eta, k, p) = -\frac{2}{3}\frac{\gamma + \sqrt{\frac{3}{4}k^2 - M\eta}}{kp} \times \log\left[\frac{k^2 + p^2 + kp - M\eta}{k^2 + p^2 - kp - M\eta}\right]. \quad (9)$$

In Fig. 4, the third virial coefficient $\bar{b}_3 \equiv b_3 - b_3^{(1)} = b_3^{(2)} + b_3^{(3)} = b_3^{(3)}$ and its derivative $T\partial_T \bar{b}_3$ are shown for ^{6}Li at temperatures around 6 μK. As expected from physical arguments presented earlier, the third virial coefficient is *universal* as $|a| \to \infty$. We find $\bar{b}_3^\infty \approx 1.05$.

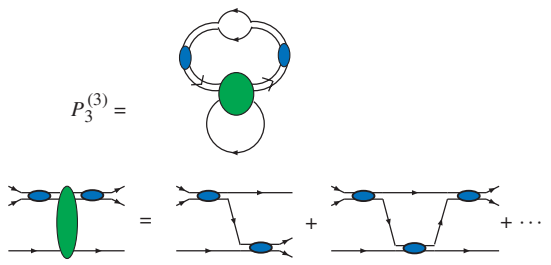

FIG. 3 (color online). 3-body contribution to the pressure $P_3^{(3)}$.

From this calculation, we see that the virial coefficients up to $\mathcal{O}(z^3)$ are not unnaturally large [numbers of $\mathcal{O}(1)$] and for $z \ll 1$, the virial expansion seems consistent. Formally, the expansion in z is equivalent to an expansion in the diluteness parameter $n\lambda^3 \ll 1$ in perturbation [7,16]:

$$n = \frac{\partial P}{\partial \mu} \approx \frac{2}{\lambda^3}[b_1 z + 2b_2 z^2 + 3b_3 z^3] + \ldots,$$
$$\Rightarrow z \approx \frac{n\lambda^3}{2} - 2b_2\left(\frac{n\lambda^3}{2}\right)^2 + (8b_2^2 - 3b_3)\left(\frac{n\lambda^3}{2}\right)^3 + \ldots, \quad (10)$$

where we used $b_1 = 1$. Defining the density n in terms of the Fermi temperature T_F, $n\lambda^3/2 = 4/(3\sqrt{\pi})(T_F/T)^{3/2}$ is the small expansion parameter for $T \gg T_F$. The energy density can be obtained from the pressure using standard thermodynamic relations $\epsilon = -P + \mu n + Ts$, where $s = \partial_T P$ is the entropy density.

At unitarity $a \to \pm\infty$, we have $\epsilon^\infty \equiv \epsilon_{\mathrm{kin}} + \epsilon_{\mathrm{int}}^\infty$ with [16]:

$$\epsilon_{\mathrm{kin}} = \frac{3}{2}Tn\left[1 + \frac{1}{2^{5/2}}\frac{n\lambda^3}{2} + \frac{n^2\lambda^6}{4}\left(\frac{1}{8} - \frac{2}{9\sqrt{3}}\right)\right],$$
$$\epsilon_{\mathrm{int}}^\infty = \frac{3}{2}Tn\frac{n\lambda^3}{2}\left[-\frac{1}{\sqrt{2}} + \frac{n\lambda^3}{2}(1 - 2\bar{b}_3^\infty)\right]. \quad (11)$$

$\epsilon_{\mathrm{int}}^\infty$ is the contribution from the interacting theory in Eq. (3).

In Fig. 5, the contributions from the second and third virial coefficient to the energy density (top graph) and the ratio $\epsilon_{\mathrm{int}}/\epsilon_{\mathrm{kin}}$ (bottom graph) are shown. Conservatively from the top graph, the range of convergence is deduced to be $T_F/T \lesssim 0.8$, where the second and the third order virial contributions are of similar size. For experiments on ^{6}Li, at temperatures around $T \sim 1.5T_F$ we expect the virial expansion to be valid. At this temperature, the effect of the

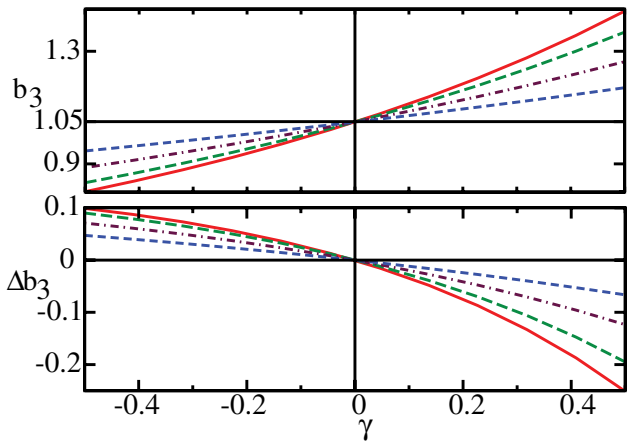

FIG. 4 (color online). Plot of $\bar{b}_3$ and its derivative $\Delta b_3 \equiv T\partial_T \bar{b}_3$ for ^{6}Li as a function of $\gamma = 1/a$ in eV. The different curves are for the same set of temperatures as in Fig. 1 and use the same notations.

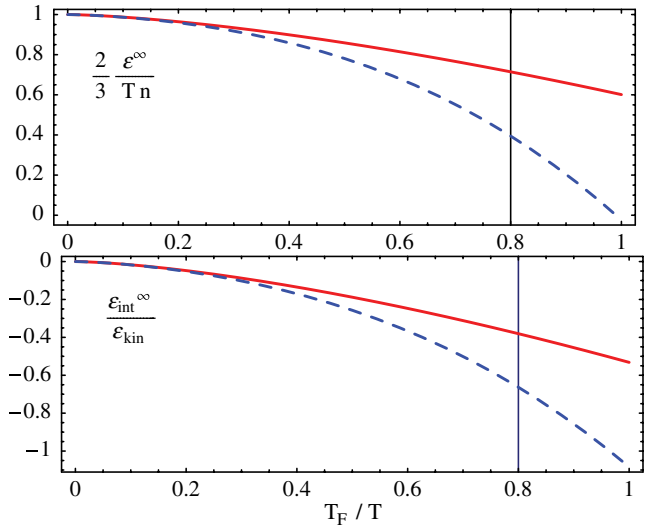

FIG. 5 (color online). The second (solid curve) and the third order (dashed curve) virial contribution in the $n\lambda^3/2$ expansion in Eq. (11).

third virial coefficient should be detectable in experimental data for dilute systems.

In conclusion, we considered a dilute spin-$\frac{1}{2}$ Fermi system near the unitarity limit $|a| \to \infty$. The thermodynamic pressure was calculated in a model independent way using effective field theory up to the third order in the virial expansion. At this order, universality was demonstrated and a range of temperature $T \gtrsim 1.25T_F$ for a fixed density (conversely a range of densities for a fixed temperature) was identified for a self-consistent application of the results. At lower temperatures, numerical lattice calculations are more appropriate [17–21]. However, at intermediate temperatures, our results could be used as checks for the lattice calculations.

The virial expansion is a useful tool for dilute systems where the microscopic physics is poorly known or difficult to calculate. The current work is of interest in atomic and nuclear physics. For example, at neutrinosphere temperatures $T \sim 5$ MeV [22,23] and densities $n \sim 10^{-3}$ fm^{-3} [6], the nuclear interaction is short ranged $R \sim 1/m_\pi \ll \lambda$, $R \ll n^{-1/3}$ with an expansion parameter $n\lambda^3 \sim 0.3$. For neutron matter, factors of $|a|/\lambda \sim 3$ have to be treated nonperturbatively giving $b_2 \approx 0.42$. Effective range r_0 corrections are expected to be important but perturbative $r_0/\lambda \sim 0.3$, in agreement with calculation of b_2 in Ref. [6]. It might be practical to include effective range corrections exactly [24,25] by incorporating it in the dimer propagator in Eq. (5). Then one would be left with small P-wave corrections $R^3/\lambda^3 \sim 1/(m_\pi\lambda)^3 \sim 0.01$. In neutron matter $a \sim -19$ fm [25], $b_3 \sim 1$ and the contribution of the third virial term to the pressure is estimated to be $n^2\lambda^6 \sim 0.09$, i.e., $\sim$10%.

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