Finger Rafting: A Generic Instability of Floating Elastic Sheets

Dominic Vella[1,*] and J. S. Wettlaufer[2,†]

[1]Institute of Theoretical Geophysics, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WA, United Kingdom

[2]Departments of Geology & Geophysics and Physics, Yale University, New Haven, Connecticut 06520-8109, USA

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Colliding ice floes are often observed to form a series of interlocking fingers. We show that this striking phenomenon is not a result of some peculiar property of ice but rather a general and robust mechanical phenomenon reproducible in the laboratory with other floating materials. We determine the theoretical relationship between the width of the resulting fingers and the material's mechanical properties and present experimental results along with field observations to support the theory. The generality of this "finger rafting" suggests that analogous processes may be responsible for creating the large-scale structures observed at the boundaries between Earth's convergent tectonic plates.

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In the natural environment floating ice is rarely at rest or alone. Ice floes are driven by wind and water stresses and hence collisions with other floes largely characterize their dynamics and rheology [1]. When two elastic sheets floating on a liquid collide, intuition leads us to expect one of two results: one sheet might be "subducted" under the other (as is observed in the Earth's crust) or the two might crush each other forming either a ridge line or a field of rubble (as observed in thick ice floes). Observations of sea and lake ice reveal that there is, in fact, a third possibility: the formation of a series of interlocking fingers that alternately ride over and under one another (as shown in Fig. 1). The fingers that are formed in this way are striking primarily because of their rectilinear features and the regular finger spacing. Despite the fact that this process, called "finger rafting" by the sea ice geophysics community, was first reported more than 50 years ago, the underlying physical mechanism and the origin of the regular finger size remain unexplained. All previous observations of finger rafting have been limited to floes in sea ice [2,3] and in fresh ice [4,5] with some evidence of finger rafting occurring in ice-tank experiments using a mixture of ethanol and water [6]. The implicit assumption has always been that finger rafting is unique to ice and so the only mechanisms suggested to explain this phenomenon have relied on the inability of sea ice to support its own weight when taken out of water—a property that is not generic amongst solids.

Using thin sheets of sealing wax [7] floating on water, we observed finger rafting simply by pushing two sheets in contact along their long edge (Fig. 2). Experimental constraints, the most important of which were our ability to accurately control the thickness of the wax and to avoid edge effects in the transverse direction, limited the number of fingers observed. However, the fingers shown in Fig. 2 have the strong rectilinear features reported of finger rafting ice and also exhibit a characteristic finger width.

We now propose an explanation for the finger rafting of floating elastic sheets, or floes for simplicity; unlike previously proposed mechanisms [2], ours does not rely on the special material properties of ice nor the generation of a swell. Instead, motivated by the naturally occurring imperfections (described below) we assume there is some small region of overlap between the two floating sheets (see Fig. 3): if a small portion of floe A overrides floe B at a point C, floe A is lifted slightly at the point C by the additional buoyancy provided by the presence of floe B. Conversely, floe B is depressed at the point C by the additional load provided by the presence of floe A. The characteristic response of a floating elastic sheet to a vertical perturbation is not monotonic decay in the far field but rather an oscillatory deflection modulated by an exponential decay. Away from the point C, therefore, both floes A and B will have an oscillatory vertical deflection, shown schematically in Fig. 3. In particular, the free edges of the two floes should have an oscillatory vertical deflection.

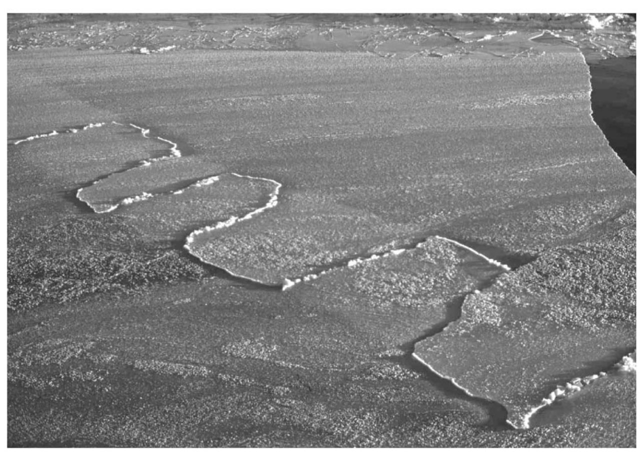

FIG. 1. The finger rafting of thin sea ice in the Amundsen Sea. Photograph courtesy of Wilford F. Weeks [22].

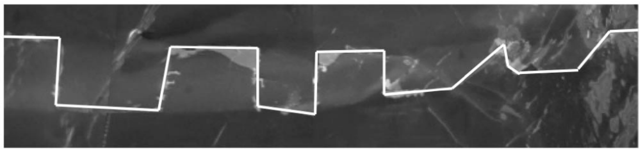

FIG. 2. Plan view of finger rafting as observed in thin wax sheets of thickness in the range 170–380 μm. The width of the field of view here is around 30 cm and the white line highlights the edge of the fingered wax sheets.

Moreover, because the initial perturbations to each of the floes are of opposite sign, these oscillations remain out of phase along the length of the free edge: crests of floe A correspond to troughs of floe B and vice versa. The free edges of the two floes are displaced vertically relative to one another, exactly half an oscillation out of phase. Upon further compression, therefore, the two floes naturally ride alternately over and under one another, thereby forming a series of interlocking fingers.

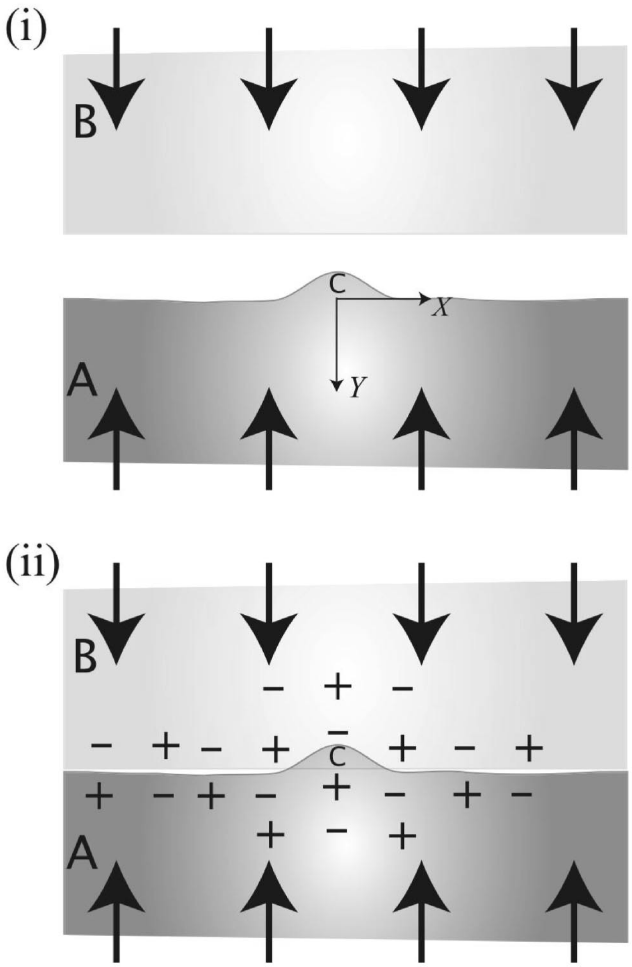

FIG. 3. Plan view of two floes A and B colliding. (i) A small protrusion in floe A leads to an overlap of the floes in a small region, C. (ii) This overlap causes oscillations in the vertical position of the floes, which decay away from C. The sign of these displacements is indicated by the $+/-$ symbols in the figure. Notice that the oscillations along the free edge are exactly out of phase in the two floes causing the two floes to alternately ride over and under each other under compression (arrows).

The mechanism we have described does not rely on any material properties that are peculiar to ice, though we do require the solid material to be able to tear to form fingers as the two floe edges are pushed past one another. This condition seems also to be satisfied by wax whereas other materials, such as aluminum foil, do not tear sufficiently easily and so cannot form these fingers.

A mechanism resembling that presented above was outlined by Fukutomi and Kusunoki [8]. They discuss finger rafting only cursorily and give a very vague presentation of a model of a point force acting on an infinite elastic sheet. However, they give no details of their calculations nor of the equations solved. Here, we present a thorough analysis of the problem.

We begin by considering the vertical force balance on an elastic sheet of thickness h and density ρ_s floating on a liquid of density ρ under the action of an externally applied pressure, $p(x, y)$. The vertical displacement of the sheet's midplane, $w(x, y)$, measured relative to the equilibrium floating depth of the sheet, $w_\infty \equiv h(1/2 - \rho_s/\rho)$, satisfies the plate equation [9]

$$\mathcal{B}\nabla^4 w + \rho g w = p(x, y), \tag{1}$$

where the bending stiffness (or bending rigidity) of the ice floe, $\mathcal{B}$, is given in terms of the Young's modulus E of the ice and its Poisson ratio, ν, by

$$\mathcal{B} \equiv \frac{Eh^3}{12(1 - \nu^2)}. \tag{2}$$

We may nondimensionalize (1) using the characteristic length,

$$\ell_* \equiv \left(\frac{\mathcal{B}}{\rho g}\right)^{1/4}, \tag{3}$$

and the characteristic pressure $\rho g \ell_*$. Using uppercase letters to denote dimensionless quantities (so that $W \equiv w/\ell_*$, $P \equiv p/\rho g \ell_*$, etc.) Equation (1) may be written

$$\nabla^4 W + W = P(X, Y). \tag{4}$$

Equation (4) also describes the response of a plate on an elastic foundation to an externally applied force. We may therefore make use of the solution for W when a point force $F = f/\rho g \ell_*^3$ is applied at the free edge of a semi-infinite elastic plate on an elastic foundation [10,11]. In particular, the vertical displacement of the edge of the sheet ($Y = 0$) a distance X from the point of application is

$$W(X, 0) = \frac{F}{\pi}\left(-\mathrm{Kei}(|X|) + \int_0^\infty A(\alpha)\cos\alpha X d\alpha\right), \tag{5}$$

where $\mathrm{Kei}(x)$ is the Kelvin function of zeroth order [12],

$$A(\alpha) \equiv \frac{k_-(k_-^2 + k_+^2 + \nu\alpha^2)}{\sqrt{\alpha^4+1}} \times \frac{2k_+^2 - (1-\nu)\alpha^2}{4k_+^2[k_-^2 + (1-\nu)\alpha^2] - (1-\nu)^2\alpha^4}, \quad (6)$$

and

$$k_\pm \equiv (\tfrac{1}{2}[\sqrt{\alpha^4+1} \pm 1])^{1/2}. \quad (7)$$

The function in (5) can be plotted numerically but what is of most interest here is the position of the zeros of $W(X)$, since these determine the regions in which the two ice floes can most easily ride over one another. The smallest X_* satisfying $W(X_*) = 0$ is $X_* \approx 4.507$ with the next root occurring at $X_* \approx 7.827$. Since the vertical displacement decays exponentially with increasing X, we take the distance between these first two roots to be that determining the finger width with the position of subsequent fingers determined once the initial fingers are in place. The wavelength of the fingering pattern is twice the finger width, so in dimensional terms we have

$$\lambda \approx 6.64\ell_*. \quad (8)$$

To test the theoretical prediction in (8), we collated field observations of finger width and ice thickness from reports of finger rafting in the literature. Figure 4 shows the typical wavelength of the fingering pattern observed in these field observations and in our experiments with wax plotted as a function of the characteristic length ℓ_*. Estimates of ℓ_* were obtained from the range of values of E and ν given in the literature for sea ice [13] and fresh ice [14]. Figure 4 demonstrates that experiments in wax show good collapse with field observations of finger rafting in ice and also with the prediction of (8). Note that $\ell_* \sim h^{3/4}$ and so there is a strong correlation between the wavelength of finger rafting and the thickness of the ice in which it is observed.

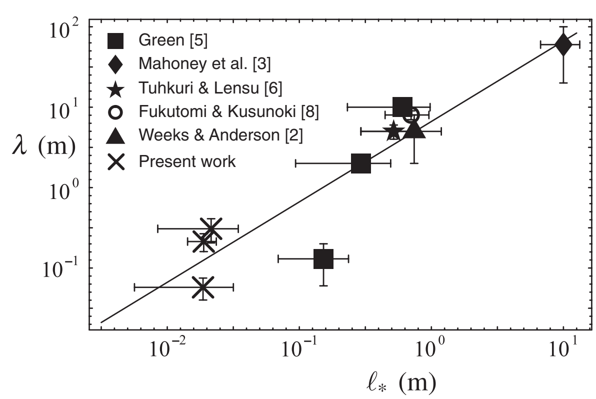

FIG. 4. Wavelength of the fingering pattern, λ, for floating sheets with differing characteristic length scales, ℓ_*. The data plotted here are a combination of those in ice (obtained from the field observations of others) and our own experiments with thin sheets of wax floating on water (denoted by $\times$). The solid line shows the theoretical prediction (8).

As already noted, the vertical displacement of the floe given in (5) decays exponentially away from the protrusion. We therefore do not expect an overlap at one place to be sufficient for finger rafting everywhere; the vertical displacement must be a reasonable fraction of the thickness for the oscillations we describe to give rise to floes running under and over one another. Rather, the formation of a series of fingers is a natural consequence of our mechanism for several reasons. First, we propose that the rafting propagates along the edge rather like a zipper: when rafting occurs in one place the displacements nearby are sufficient to cause finger rafting there too and so on. This wave of rafting should travel at the speed of gravity waves in water covered with an elastic sheet. The phase speed, c, of these waves [15] depends on their wave number, k. In our wax experiments, the speed of the waves with wave number ℓ_*^{-1} is typically around 0.5 ms^{-1}, making this zippering unobservable within the scope of the technology we employed. For sea ice of thickness 10 cm, this wave speed is on the order of 5 ms^{-1}. Second, we note that finger rafting often occurs after a single sheet fails under buckling and produces two daughter sheets, each with ragged edges. These ragged edges are a ready supply of protrusions each of which can evolve according to the zipper scenario. This, along with inhomogeneities in the ice thickness, readily explains many of the observations in which the finger sizes vary.

Contrary to what may have been expected on the basis of the available literature, we have shown that the phenomenon of finger rafting is a generic mechanical process. As such, not only does the analysis explain the origin of the ubiquitous and appealing features in the polar ice cover, but extrapolating the results shown in Fig. 4 to large and small scales also presents intriguing, even if speculative, possibilities. On the small scale our mechanism may contribute to the goals of refining methodologies in micromachining, with applications ranging from engineering to biology [16]. On the large scale, our system of two elastic sheets floating on a viscous liquid may serve as a model for both plate tectonics and the mechanics of icy moons. In particular, modifications of our thinking may lead to predictions of relevance to similar finger like features in these situations. Indeed, we note that the Scotia and Caribbean tectonic plates are both moving eastwards as the Nazca plate is subducted beneath the South American plate [17]. Both of these plates are elongated in their direction of motion and have rectilinear boundaries: two features characteristic of finger rafting. The fact that these two plates also have similar dimensions suggests that they could perhaps be the remnants of finger rafting that may have occurred as the Nazca plate collided with the South American plate. In fact, the similarity between the geometry of these plates and that produced by finger rafting in ice has been noticed before [18]. Here we go beyond this analogy and suggest that the magnitude of the "finger width'' in these tectonic plates (~500 km) is consistent with our elastic mechanism. Taking the typical plate thickness to lie in the range 30–100 km and elastic properties from [19], Eq. (8) gives predicted finger widths in the range 170–430 km. In this situation, we expect site specific details (most notably the unequal plate thicknesses and the presence of a vertical temperature gradient) to complicate our physical picture somewhat. However, we also note that finger rafting is consistent with the geological history of the Scotia Arc, which points to there having been a continuous Andean-West Antarctic Cordillera more than 20×10^6 years ago [20].

Finally, our ideas may also be applicable to some of the very regular surface features observed on icy moons. Earlier work [21] has suggested that the ice thickness on these moons must be below some critical threshold for cycloidal cracks to form. We have shown that the surface patterns of finger rafted objects is closely correlated to their thickness and material properties potentially allowing the ice thickness of these moons to be determined more precisely.

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*Electronic address: d.vella@damtp.cam.ac.uk
†Electronic address: john.wettlaufer@yale.edu

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