Localization in a Quantum Spin Hall System

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The localization problem of electronic states in a two-dimensional quantum spin Hall system (that is, a symplectic ensemble with topological term) is studied by the transfer matrix method. The phase diagram in the plane of energy and disorder strength is exposed, and demonstrates "levitation" and "pair annihilation" of the domains of extended states analogous to that of the integer quantum Hall system. The critical exponent ν for the divergence of the localization length is estimated as $\nu \cong 1.6$, which is distinct from both exponents pertaining to the conventional symplectic and the unitary quantum Hall systems. Our analysis strongly suggests a different universality class related to the topology of the pertinent system.

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It is well established that localization properties of electron wave functions in disordered systems depend only on spatial dimensionality, symmetry, and on the presence or absence of topological terms in the Lagrangian, and are independent of the details of the model. This universality has been an issue of intensive interests over decades. The aim of this work is to shed light on electron localization and Anderson transition for a system which exhibits a very robust metallic region as well as a new kind of topological properties. This is achieved by studying localization properties of two-dimensional electronic systems exhibiting the spin Hall effect (SHE—a generation of spin current perpendicular to an applied electric field [1–9]). The fundamental feature of the SHE is the underlying topological structure of Bloch wave functions in systems with time-reversal (TR) symmetry. Recall that for a TR violating system such as the unitary ensemble in two dimensions, the occurrence of the edge channel transport represented by the topological structure [10,11] (associated with gauge invariance and conserved charge current) dramatically affects its properties: it implies the quantization of the Hall conductance and leads to a delocalization transition in an otherwise localized system. On the other hand, *for TR invariant systems*, recently introduced Z_2 topological numbers [12–16] are not directly associated with a conserved current, and their influence on localization properties has not yet been elucidated. It should be stressed that the topological number Z_2 [13] has a simple and transparent physical content: It indicates the parity of the number of Kramers degenerate edge modes in an open system. Accordingly, SHE systems may be divided into: (1) *spin Hall insulators* (SHI) [17]—band insulators showing nonzero spin Hall conductivity such that Z_2 is even. Consequently, edge modes annihilate each other and do not close the gap [18]. (2) *quantum spin Hall* phases (QSH—see also Ref. [19]), are characterized by a bulk electronic band gap with Z_2 odd, thereby supporting the transport of charge and spin in gapless edge states. The stability of the edge modes against the backward scattering in this case is guaranteed by Kramers' theorem [20,21]. The physical realization of the QSH phase has been proposed for several explicit systems such as two-dimensional (i) graphene [12,13], (ii) surface state of Bi [22], (iii) quantum well structure of CdTe/HgTe/CdTe [23], and three-dimensional (iv) $Bi_{1-x}Sb_x$, (v) α-Sn and HgTe under pressure [24].

In this Letter, we study a model for disordered graphene [12,13] to see how the occurrence of gapless edge states in the QSH phase (odd Z_2) affects the Anderson localization. From the universality of the Anderson metal-insulator localization transition, it is expected that this results applies also to other QSH systems, i.e., two-dimensional symplectic ensemble with nontrivial Z_2 topological number. In contrast with the situation encountered for the unitary ensemble (where topological term induces a transition at a single energy), we encounter here a symplectic system with a well-defined region of metallic states albeit with a nontrivial topological structure. Using the transfer matrix method [25], the phase diagram in the plane of disorder strength and energy is revealed, which manifests the features of *levitation* and *pair annihilation* of extended states similar to the unitary QHE case with the Chern number $C_{U(1)} \neq 0$, albeit with finite energy width of the extended region. Finite size scaling analysis of the localization or delocalization transition yields an exponent $\nu \cong 1.6$ for the divergence of the localization length, which is distinct from both that of the symplectic ($\nu \cong 2.73$) [26] and that of the unitary QHE ($\nu \cong 2.33$) [27] universality classes. This strongly suggests that the symplectic model with odd Z_2 number belongs to a new universality class from the viewpoint of the Anderson localization.

We study the following Hamiltonian for the disordered graphene [12,13]:

$$H = \sum_{\langle ij \rangle} c_i^\dagger [t + i\lambda_R (\boldsymbol{\sigma} \times \hat{\boldsymbol{d}}_{ij})_z] c_j + \sum_{\langle ij \rangle'} c_i^\dagger (t' + i\lambda_{\rm SO} \nu_{ij} \sigma_z) c_j + \sum_i c_i^\dagger (\lambda_v \eta_i + w_i + h_x \sigma_x) c_i. \quad (1)$$

Here $c_i^{(\dagger)}$ is the spinor annihilation (creation) operator, $\boldsymbol{\sigma}$ is the set of the Pauli matrices, t is the conventional hopping energy between nearest neighbor sites $\langle ij \rangle$ (we use $t = 1$ as the unit of energy), λ_R is the Rashba spin-orbit interaction strength, and $\hat{\boldsymbol{d}}_{ij}$ is the unit vector connecting $\langle ij \rangle$. The constant t' is the conventional hopping energy between second nearest neighbor sites $\langle ij \rangle'$. Here it is introduced just for assuring stability of the numerical analysis by the transfer matrix method and is fixed at the small value 0.01. Moreover, $\lambda_{\rm SO}$ represents the spin-orbit interaction strength with $\nu_{ij} = (2/\sqrt{3})(\hat{\boldsymbol{d}}_1 \times \hat{\boldsymbol{d}}_2) = \pm 1$, where $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$ are unit vectors along the two bonds connecting $\langle ij \rangle'$, while λ_v is the alternation of the site energies between the A and B sublattices ($\eta_i = \pm 1$). The random potential w_i is uniformly distributed between $-W/2$ and $W/2$. Finally, the last term represents the magnetic field along the x direction, which breaks TR symmetry. The phase diagram of this model without the random potential w_i and the magnetic field h_x has been already displayed in the inset of Fig. 1 in Ref. [12]. A recent work introduces another topological number, i.e., the spin Chern number $C_{\rm sc}$, associated with twisted (spin-dependent) boundary conditions [15]. It is pointed out that $C_{\rm sc}$ is related to the Z_2 classification as $Z_2 = (C_{\rm sc} {\rm mod} 4)/2$ [16]. In the plane $\lambda_v/\lambda_{\rm SO}$-$\lambda_R/\lambda_{\rm SO}$, there is a finite domain of the QSH state with $Z_2 = 1$, $C_{\rm sc} = 2$, bounded by a curve where the gap closes. Outside this domain, the gap opens up again and the system becomes the usual SHI with $Z_2 = 0$, $C_{\rm sc} = 0$. Note that for $\lambda_R = 0$, the system is decoupled into two independent unitary subsystems [28]. For $\lambda_v < \lambda_v^c$, each unitary model has $C_{U(1)}$'s of opposite signs for the valence and conduction bands, and the way of distribution of $C_{U(1)}$'s is opposite for each unitary model. Thus the total $C_{U(1)}$ of valence bands is zero, but $C_{\rm sc} = 2$. In the case of $\lambda_v > \lambda_v^c$, $C_{U(1)}$'s of each band vanishes, and $C_{\rm sc}$ is also zero. For finite λ_R, these two unitary models are hybridized and become a symplectic model. $C_{\rm sc}$ of this hybridized system is quantized as $C_{\rm sc} = 2$ in the domain of the QSH state, while it vanishes in the domain of the usual SHI. With a finite magnetic field h_x, similar hybridization occurs, but this breaks TR symmetry, and the model belongs to the topologically trivial unitary class, where $C_{U(1)}$ of each band is zero. Interestingly, in the clean limit, $C_{\rm sc}$ of this unitary model is still quantized, i.e., $C_{\rm sc} = 2$, while each pair of edge states opens a gap due to the hybridization by h_x. (Note that $C_{\rm sc}$ is well defined even in TR breaking systems.)

The localization length $\lambda_M(W, E)$ of a long tube of $2M$-site circumference is calculated at energy E and disorder strength W by the transfer matrix method [25]. The transfer matrices are iteratively multiplied to wave functions (vectors), with the orthonormalizations at every nth ($n \leq 10$) step to keep the information on the eigenvalue with modulus closest to unity. The localization length can be obtained from this eigenvalue. The M dependence of the renormalized localization length $\Lambda_M(W, E) = \lambda_M(W, E)/M$ determines the localization or delocalization properties of the wave functions at energy E.

Figures 1(a)–1(c), display $\Lambda_M(W, E)$ up to $M = 24$ for several values of W as functions of E for disordered QSH system at $h_x = 0$ (symplectic ensemble). The case $\lambda_R = 0$ has been studied in the context of quantized anomalous Hall effect [28], and the two isolated extended states are

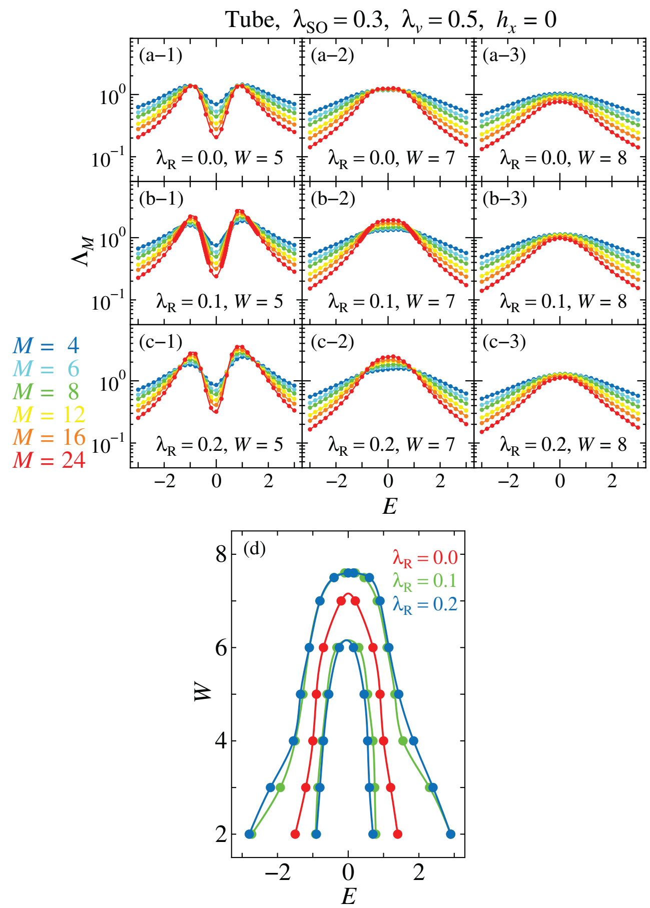

FIG. 1 (color). Renormalized localization length $\Lambda_M(W, E) = \lambda_M(W, E)/M$ for QSH with $\lambda_R = 0$ (a-1),(a-2),(a-3), $\lambda_R = 0.1$ (b-1),(b-2),(b-3), and $\lambda_R = 0.2$ (c-1),(c-2),(c-3). The disorder strength W is increased from left to right as $W = 5.0$ (a),(b),(c-1), $W = 7.0$ (a),(b),(c-2), and $W = 8.0$ (a),(b),(c-3). The other parameters are fixed as $\lambda_{\rm SO} = 0.3$, $\lambda_v = 0.5$, and $h_x = 0$. (d) A localization or delocalization phase diagram obtained in the plane of energy E and disorder strength W. The red curve is the energy of the isolated extended states for $\lambda_R = 0$, while the green and blue curves are the boundary of the energy region of the extended states for $\lambda_R = 0.1$ and $\lambda_R = 0.2$.

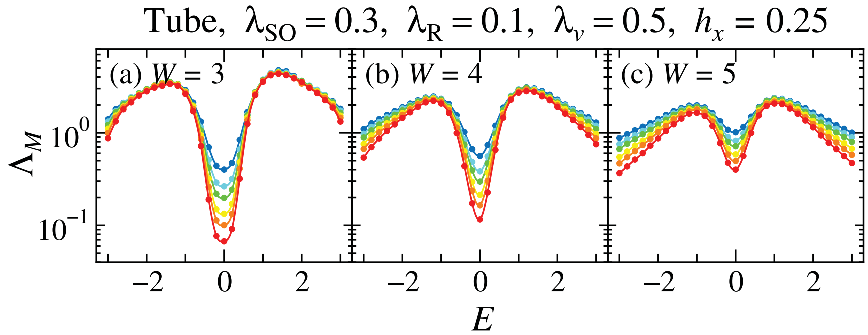

FIG. 2 (color). Renormalized localization length $\Lambda_M(W, E) = \lambda_M(W, E)/M$ for the unitary model with $h_x = 0.25$ at (a) $W = 3.0$, (b) $W = 4.0$, (c) $W = 5.0$. The other parameters are $\lambda_{SO} = 0.3$, $\lambda_R = 0.1$, and $\lambda_v = 0.5$. All the states are already localized since the U(1) Chern numbers for up and down spin bands cancel out.

identified by M-independent $\Lambda_M(W, E)$. There are two energies at which extended states show up. As they merge together the extended states disappear. This is consistent with the scenario of levitation and pair annihilation of two first Chern numbers $C_{U(1)}$ having opposite signs. With finite λ_R, the isolated extended states turn into finite energy region of extended states, the width of which increases with λ_R. Note that these two regions of extended states approach as the disorder W increases. The gap in the density of states disappears already for W larger than $\sim$3, but still the two regions are separated. When W is further increased, these two energy regions of extended states merge into one region, and eventually disappear. We have also checked that, in the ribbon geometry, there appear extended gapless edge states even when there are no extended bulk states in the middle energy region. Based on these results, we draw in Fig. 1(d) a phase diagram depicting the location of extended states in the E-W plane. The red curve for $\lambda_R = 0$ represents the trajectories of the isolated extended states in the unitary (QHE) case, while the phase boundaries between the localized and extended states are given by green curves for $\lambda_R = 0.1$ and by blue ones for $\lambda_R = 0.2$.

Let us confront some other cases with those discussed in Fig. 1. Figures 2(a)–2(c) display the curves $\Lambda_M(W, E)$ for the unitary model with $h_x = 0.25$ at (a) $W = 3.0$, (b) $W = 4.0$, and (c) $W = 5.0$. In this case, the extended states have already disappeared since $C_{U(1)}$ is zero for each of the split bands. Therefore the model is reduced to the trivial unitary class, where all states in two dimensions are localized with any finite amount of disorder [29]. (It is also confirmed that the gapful edge states in the ribbon geometry are localized.) However, it should be noted that C_{sc} for this system is quantized as $C_{sc} = 2$ in the clean limit. This means that, in TR breaking systems, finite C_{sc} does not protect extended states and that protection would be closely related to Kramers' theorem. One might think that the difference is due solely to symmetry, i.e., unitary vs symplectic classes, and not to the topological property of the QSH state. It is therefore important to compare with the simple symplectic model which belongs to the trivial Z_2 classification. In our model, the SHI corresponds to this case, and Fig. 3 shows $\Lambda_M(W, E)$ for $\lambda_{SO} = 0.05$, $\lambda_R = 0.3$, $\lambda_v = 1.0$ at (a) $W = 2.6$, (b) $W = 3.0$, and (c) $W = 3.4$, respectively. It is evident here that the extended states disappear with much weaker disorder strength, and the two energy regions of extended states disappear without merging into a single one.

The above two cases, i.e., Figs. 2 and 3 strongly suggest that the localization behavior of QSH system in Fig. 1 is deeply influenced by the nontrivial topological aspect, and is distinguished from that of the usual symplectic class. In order to substantiate this expectation, we have studied the critical property of the localization or delocalization transition of the QSH system.

Figure 4 summarizes the scaling analysis by displaying $\Lambda_M(W, E) = f((E - E_c)M^{1/\nu})$ at $W = 5.0$ with ν being the critical exponent for the divergence of the localization length. Data for various E and M (up to $M = 24$) are included and their collapse on a single curve indicates a reasonable one parameter scaling behavior, which simultaneously determines $\nu = 1.61 \pm 0.10$. We have also studied transition at higher disorder $W = 7.0$, and found $\nu = 1.61 \pm 0.10$. This exponent should be confronted with that of the standard symplectic universality class $\nu \cong 2.73$ [26], and that of the unitary model at strong magnetic field

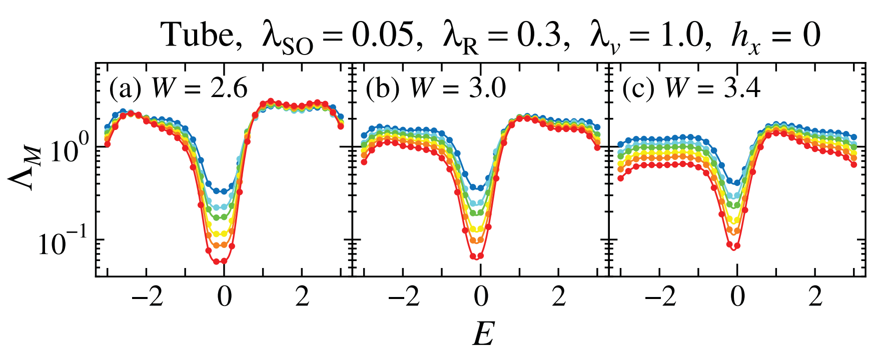

FIG. 3 (color). Renormalized localization length $\Lambda_M(W, E) = \lambda_M(W, E)/M$ for SHI with $\lambda_{SO} = 0.05$, $\lambda_R = 0.3$, and $\lambda_v = 1.0$ at (a) $W = 2.6$, (b) $W = 3.0$, and (c) $W = 3.4$.

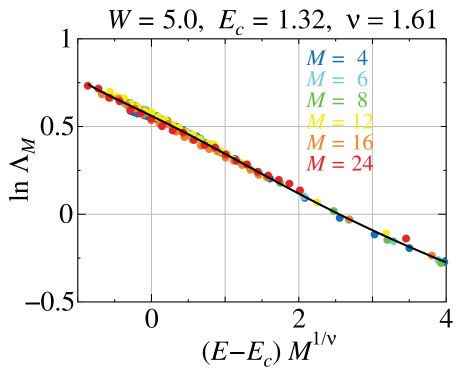

FIG. 4 (color). Scaling plot of the renormalized localization length $\Lambda_M(W, E)$ of Fig. 1(b-1) as a function of $(E - E_c)M^{1/\nu}$ for different energies E and $M \leq 24$. The parameters are $\lambda_{SO} = 0.3$, $\lambda_R = 0.1$, $\lambda_\upsilon = 0.5$ $h_x = 0$, and $W = 5.0$. The critical energy and exponent are estimated as $E_c = 1.32 \pm 0.005$, $\nu = 1.61 \pm 0.10$, respectively.

(with finite $C_{U(1)}$) $\nu \cong 2.33$ [27]. The result $\nu \cong 1.6$ obtained here is clearly distinct from both of these values, and suggests a new universality class for the symplectic ensemble with a nonzero topological structure. Intuitively, it is anticipated that the localization problem should be influenced by the odd Z_2 number. The construction of an effective field theory for this class is left for future investigations.

In summary, we have studied the localization or delocalization problem of the QSH system, which represents a special class of symplectic ensembles with nontrivial topological properties. The phase diagram shows levitation and pair annihilation of the two energy regions of extended states, analogous to that of the unitary model with finite Chern number (the integer quantum Hall effect). The critical exponent ν for the localization or delocalization transition is estimated as $\nu \cong 1.6$, which is distinct from that of the standard symplectic class ($\nu \cong 2.73$) and that of the unitary class with nonzero Chern number ($\nu \cong 2.33$). This strongly suggests that the QSH system belongs to a new universality class characterized by a topological index such as the Z_2 index.

Finally, it is worth mentioning here that localization of electronic wave functions in disordered graphene has recently been studied [30–35]. In these works the electrons are assumed to be spinless and therefore the underlying ensemble is generically the orthogonal one where all states are localized. Yet, due to the special structure of the graphene lattice and the Dirac spectrum near the K, K' points, the theory of weak localization developed for this system appears to be extremely rich; i.e., it displays the crossover from the symplectic class to the orthogonal one.

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