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Conformal Field Theory of Composite Fermions

T. H. Hansson,[1] C.-C. Chang,[2] J. K. Jain,[2] and S. Viefers[3]

[1]*Department of Physics, Stockholm University AlbaNova University Center, SE-106 91 Stockholm, Sweden*
[2]*Physics Department, 104 Davey Lab, The Pennsylvania State University, University Park, Pennsylvania 16802, USA*
[3]*Department of Physics, University of Oslo, P.O. Box 1048 Blindern, 0316 Oslo, Norway*
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We show that the quantum Hall wave functions for the ground states in the Jain series $\nu = n/(2np + 1)$ can be *exactly* expressed in terms of correlation functions of local vertex operators V_n corresponding to composite fermions in the nth composite-fermion (CF) Landau level. This allows for the powerful mathematics of conformal field theory to be applied to the successful CF phenomenology. Quasiparticle and quasihole states are expressed as correlators of anyonic operators with fractional (local) charge, allowing a simple algebraic understanding of their topological properties that are not manifest in the CF wave functions. Moreover, our construction shows how the states in the $\nu = n/(2np + 1)$ Jain sequence may be interpreted as condensates of quasiparticles.

In an intriguing line of development, many of the fractional quantum Hall (FQH) effect wave functions, both ground states and the associated quasihole states, have been expressed as certain correlators in two-dimensional conformal field theories (CFTs). Among these are the Laughlin states [1] at Landau level fillings $\nu = 1/m$ (m being an odd integer) and the Moore-Read Pfaffian wave function [2] at $\nu = 1/2$. While there is no fundamental understanding of why the quantum mechanical wave functions of a 2D electron gas in the lowest Landau level (LLL) should bear any relation to correlation functions of vertex operators in a 2D Euclidian CFT, many arguments for such a connection were given in the pioneering paper by Moore and Read [2] and discussed and extended in several subsequent works [3].

The charged excitations in the FQH systems, in general, have a fractional charge [1] and exhibit fractional statistics [4]—they are anyons [5]. These are a manifestation of subtle long range quantum correlations that characterize the FQH states, which are said to be topologically ordered. When several such excitations form composites, their charges and statistical properties can be deduced from their constituents using *local* rules. The fractional excitations as well as these local rules of "fusion" and the related "braiding" (i.e., statistics) properties of the FQH states find their counterpart in CFTs. The excitations are represented by operators, the fusion corresponds to their operator product expansion (OPE), and the braiding rules follow from certain algebraic properties (monodromies) of their correlation functions (or conformal blocks).

There are, however, important limitations to the CFT approach. A connection between CFT and quasiparticle (as opposed to quasihole) excitations at $\nu = 1/m$ has so far been lacking [3]. Furthermore, in spite of interesting progress [6], no CFT expressions have so far been established for the experimentally prominent FQH states in the

Jain series at filling fractions $\nu = n/(2np + 1)$ and their quasiparticle or quasihole excitations.

The composite-fermion (CF) formalism provides for a unified description of the Jain ground states and their quasiparticle and quasihole excitations. A composite fermion is an electron bound to $2p$ units of vorticity, and the ground states are n filled CF Landau levels. The phenomenology based on composite fermions has been successful, both in comparison with experiments and with various exact numerical studies [7].

CF wave functions for quasiparticles and quasiholes are obtained by adding a composite fermion to, or removing it from, a CF Landau level. Although phrased in a single particle language, this is a nonlocal operation in terms of electrons and qualitatively different from the corresponding local [8] process in an integral QH state. Thus, simplistic analogies can lead to qualitatively wrong conclusions. While fractional charge and fractional braiding statistics can be derived from the pertinent CF wave functions, these topological properties are not manifest.

In this Letter, we present an exact representation of the CF wave functions in the Jain series as correlators of vertex operators in a CFT involving n compactified boson fields [9]. Quasiparticle and quasihole excitations are described by operators that explicitly obey anyonic statistics and have fractional charge, thus extending the CFT connection between topological properties of excitations and algebraic properties of vertex operators from electrons and quasiholes to include quasiparticles. The CFTs describing the Jain states also explicitly give the effective theory within the K-matrix approach [10] and thus the pertinent edge theory. This gives further evidence for the conjecture in Ref. [2] that the edge dynamics is given by the same CFT that gives the bulk wave functions.

We first recall how to construct the ground state and quasihole wave functions at the Laughlin fractions

$\nu = 1/m$. Following Ref. [2], we define the vertex operators:

$$V_1(z) = e^{i\sqrt{m}\varphi_1(z)}, \tag{1}$$

$$H_{1/m}(\eta) = e^{(i/\sqrt{m})\varphi_1(\eta)}, \tag{2}$$

with $z = x + iy$. The field φ_1 is a free massless boson normalized as to have the (holomorphic) two point function $\langle \varphi_1(z)\varphi_1(w)\rangle = -\ln(z - w)$. φ_1 is compactifed on a circle of radius $R^2 = m$ so the (holomorphic) $U(1)$ charge density operator is $J(z) = (i/\sqrt{m})\partial\varphi_1(z)$, with $\partial = \partial/\partial z$. The corresponding $U(1)$ charge $\mathcal{Q} = (1/\sqrt{m})\frac{1}{2\pi} \times \oint dz\partial\varphi_1(z)$, which can be thought of as vorticity, can be read directly from the commutators $[\mathcal{Q}, V_1(z)] = V_1(z)$ and $[\mathcal{Q}, H_{1/m}(\eta)] = \frac{1}{m}H_{1/m}(\eta)$ and equals 1 for the electron and $1/m$ for the hole. Following the basic idea of Read [11], the total *excess electric charge* associated with a vertex operator $Q_{el} = e(Q - \delta n)$ gets contributions from both the electrons actually added δn and the change in vorticity Q, corresponding to a depletion or contraction of the electron liquid. If the argument of the vertex operator is an electron coordinate z_i, one electron is added, while no electron is added if the argument is a hole coordinate η_i.

The $\nu = 1/m$ Laughlin ground state wave function $\Psi_L(\{z_k\}) \equiv \Psi_L = \prod_{j<k}(z_j - z_k)^m$ can be written [12]

$$\Psi_L = \langle V_1(z_1)V_1(z_2)\cdots V_1(z_{N-1})V_1(z_N)\rangle. \tag{3}$$

The wave function for a collection of Laughlin quasiholes is given by similar expressions containing insertions of the hole operator $H_{1/m}$.

The most natural guess [2] for a quasiparticle operator would be to change the sign in the exponent in the hole operator of (2). This, however, introduces singular terms $\sim \prod_i(z_i - \eta)^{-1}$ in the electronic wave function. Inspired by the CF wave functions, we instead define a quasiparticle operator $P_{1/m}(z)$ by modifying one of the electron operators to have $U(1)$ charge $(1 - 1/m)$. The excess electric charge associated with such a modification is the difference between the charges of the operators V_1 and $P_{1/m}$, i.e., $\Delta Q_{el} = e[(1 - 1/m) - 1] = -e/m$ appropriate for a Laughlin quasiparticle. The modified electron operator is given by

$$P_{1/m}(z) = \partial e^{i[\sqrt{m}-(1/\sqrt{m})]\varphi_1(z)}. \tag{4}$$

The wave function for a single quasiparticle with angular momentum l takes the form (from now on, we will suppress the Gaussian factor $\exp\{-\sum_i |z_i|^2/(4\ell^2)\}$, where $\ell^2 = \hbar/eB$):

$$\Psi_{1qp}^{(l)} = \mathcal{A}\left\{z_1^l e^{-|z_1|^2/4m\ell^2}\left\langle P_{1/m}(z_1)\prod_{i=2}^N V_1(z_i)\right\rangle\right\}$$

$$= \sum_i (-1)^i z_i^l \prod_{j<k}^{(i)}(z_j - z_k)^m \partial_i \prod_l^{(i)}(z_l - z_i)^{m-1}, \tag{5}$$

where $\mathcal{A}$ denotes antisymmetrization of the coordinates,

and the superscript of $\prod$ indicates indices omitted in the product (e.g., $\prod_i^{(j)} = \prod_{i=1 \atop i\neq j}^N$ and $\prod_{i<j}^{(kl)} = \prod_{i<j \atop i,j\neq k,l}^N$). The last product in (5) comes from contracting the operators $P_{1/m}$ and V_1; since $(m - 1)$ is even, the derivative is important to ensure that antisymmetrization does not annihilate the state. Put differently, contracting the electron liquid around the point z_i amounts to having a leading short distance part in the wave function $\sim(z_l - z_i)^{m-2}$, as seen from the OPE $P_{1/m}(z)V_1(w) \sim (z - w)^{m-2}$, where the power changes from $(m - 1)$ to $(m - 2)$ because of the derivative.

The wave function in (5) is *identical* to the corresponding CF result [Eq. (5) of Ref. [13]] and, thus, has a good variational energy and the correct fractional charge. A localized quasiparticle state can be constructed as a coherent superposition of the angular momentum eigenstates of (5). Note that our construction is entirely within the LLL— no projection is needed [14]. The operator $P_{1/m}(z)$ is not fermionic, as can be seen from the OPE $P_{1/m}(z)P_{1/m}(w) \sim (z - w)^{[m-4+(1+m)]}$. The precise connection to composite fermions is clarified below.

The quasiparticle wave function of (5) has a different character than those written earlier for the ground and the quasihole states, in that it is a sum over correlators and that it involves a prefactor. The factor z^l gives the correct angular momentum, and the exponential factor is chosen to give the correct LLL electronic wave function. It is suggestive that these prefactors precisely constitute the LLL wave function $f_1 = z_i^l e^{-|z_i|^2/4m\ell^2}$ for a charge e/m particle in the LLL. Although we have no fundamental derivation of this, we find below a similar interpretation in the case of several quasiparticles, where their anyonic nature is also manifest.

A natural generalization to two quasiparticles is

$$\Psi_{2qp} = \mathcal{A}\left\{f_2(z_1, z_2)\left\langle P_{1/m}(z_1)P_{1/m}(z_2)\prod_{i=3}^N V_1(z_i)\right\rangle\right\}. \tag{6}$$

The correlator gives a factor $\partial_i\partial_j(z_i - z_j)^{m-2+1/m}$, so to get an analytic electron wave function with the limiting behavior $\sim(z_i - z_j)^{m-1+l}$ with $l \geq 1$ and odd, we should take $f_2(z_i, z_j) = g(Z_{ij})(z_i - z_j)^{1+l-(1/m)}e^{-(|z_i|^2+|z_j|^2)/4m\ell^2}$, where $Z_{ij} = (z_i + z_j)/2$. This is precisely the expected wave function for two fractionally charged anyons in the LLL. Evaluating the correlator gives the explicit CFT wave function for two quasiparticles in relative angular momentum l and center of mass angular momentum L:

$$\Psi_{2qp} = \sum_{i<j}(-1)^{i+j}Z_{ij}^L z_{ij}^{1+l-(1/m)}\partial_i\partial_j z_{ij}^{m-2+(1/m)}\prod_k^{(j)} z_{ki}^{m-1}$$

$$\times \prod_p^{(i)} z_{pj}^{m-1}\prod_{q<r}^{(ij)} z_{qr}^m, \tag{7}$$

where $z_{ij} = z_i - z_j$ and the derivatives act on the whole

TABLE I. The expectation values of the Coulomb Hamiltonian with respect to the two-quasiparticle CFT wave functions in (7) (E_{CFT}), with $L = 0$ and $l = 1$, and the corresponding CF wave function in Eq. (14) of Ref. [13] (E_{CF}). The energies are quoted in units of e^2/ℓ, $\ell = \sqrt{\hbar c/eB}$. The last column gives the overlap between the two wave functions. N is the number of particles.

N	E_{CFT}	E_{CF}	$\langle \Psi_{2qp} | \Psi_{2qp}^{CF} \rangle$
10	7.766 19(62)	7.766 00(62)	0.999 930 1(2)
20	24.1403(19)	24.1402(19)	0.999 927 4(4)
30	46.3258(18)	46.3257(18)	0.999 927 4(3)
40	73.2339(17)	73.2339(17)	0.999 926 6(2)

expression to their right. The first nontrivial test of our construction is whether (6) produces accurate wave functions. To investigate that, we compare this CFT wave function with the standard CF wave function [13]. The latter is similar in structure but not identical; it can be obtained from (7) by replacing $z_{ij}^{1+l-(1/m)} \partial_i \partial_j z_{ij}^{m-2+(1/m)}$ with $z_{ij}^l \partial_i \partial_j z_{ij}^{m-1}$. The comparisons, using the Metropolis Monte Carlo algorithm, for a quasiparticle pair with $L = 0$ and $l = 1$ are summarized in Table I and Fig. 1 [15]. The almost perfect agreement between the two wave functions is surprising.

The properties of the operator $P_{1/m}$, as well as the analytic form of the wave function in (6), strongly suggest that the CFT quasiparticle has good statistics properties, which is also all but guaranteed by the near identity to the corresponding CF wave function. However, an analytical derivation of the quasiparticle statistics is both of intrinsic interest and a demonstration of the power of the CFT approach. In the case of quasiholes, charge and statistics
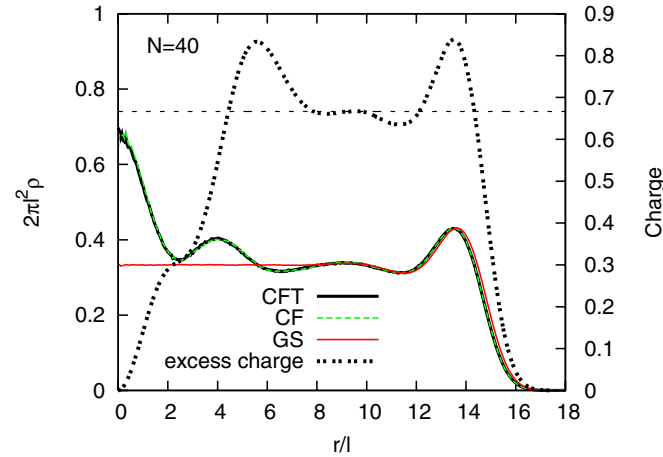

FIG. 1 (color online). The density profile for the CFT state in (7) which represents two quasiparticles at $\nu = 1/3$, along with the standard CF state. The total number of electrons is $N = 40$. The excess charge (above the $1/3$ ground state labeled by GS) inside the disk of radius r has a clear plateau at $2/3$ outside the region containing the two quasiparticles and also away from the edge. The radius r is shown in units of the magnetic length and the density ρ in units of the $\nu = 1$ density $(2\pi l^2)^{-1}$.

can be extracted from Berry phases evaluated using the Laughlin plasma analogy or a generalization thereof based on properties of CFT correlators, proposed by Nayak and Wilczek [16]. Although the quasiparticle case is less straightforward, since the electronic wave functions involve sums over correlators, analytical calculations of Berry phases are possible [9].

We now generalize to M quasiparticles at $\nu = 1/m$. We consider a maximum density circular droplet obtained by putting all of the quasiparticle pairs in their lowest allowed relative angular momentum and with zero angular momentum for the center of mass. For simplicity, we now specialize to $m = 3$. The resulting wave function

$$\Psi_{Mqp} = \sum_{i_1 < i_2 \dots < i_M} (-1)^{\sum_k i_k} \prod_{k<l}^M (z_{i_k} - z_{i_l})^{5/3} e^{-\sum_i^M |z_i|^2/12\ell^2}$$
$$\times \left\langle P_{1/3}(z_{i_1}) \dots P_{1/3}(z_{i_M}) \prod_{j \notin |\{i_1 \dots i_M\}}^N V_1(z_j) \right\rangle \quad (8)$$

again differs from the CF wave function only in the ordering of the derivatives and the Jastrow factors and can be expected to be accurate.

This suggests an exact representation of the CF wave functions as CFT correlators by introducing several bosonic fields. The factor $\prod_{k<l}^M (z_{i_k} - z_{i_l})^{5/3} \times \exp(-\sum_i^M |z_i|^2/12\ell^2)$ in (8), which has the interpretation of an M-quasiparticle wave function $f(z_i)$ of a maximum density droplet of anyons in the LLL, can be expressed as a correlator of vertex operators of a second free bosonic field φ_2. Define

$$V_2(z) = \partial e^{i(2/\sqrt{3})\varphi_1(z)} e^{i(\sqrt{5/3})\varphi_2(z)}, \quad (9)$$

and consider the correlator

$$\Psi_{Mqp} = \sum_{i_1 < i_2 \dots < i_M} (-1)^{\sum_k^M i_k}$$
$$\times \left\langle V_2(z_{i_1}) \dots V_2(z_{i_M}) \prod_{j \notin |\{i_1 \dots i_M\}} V_1(z_j) \right\rangle \quad (10)$$

of M V_2's and $(N - M)$ V_1's. If we had chosen $V_2(z) = e^{i(\sqrt{5/3})\varphi_2(z)} \partial e^{i(2/\sqrt{3})\varphi_1(z)}$, (10) would have reproduced (8). With the choice in (9), we obtain instead

$$\Psi_{Mqp} = \sum_{i_1 < i_2 \dots < i_M} (-1)^{\sum_m^M i_m} \partial_{i_1} \cdots \partial_{i_M} \prod_{i_l < i_m} (z_{i_l} - z_{i_m})$$
$$\times \prod_{\bar{i}_l < \bar{i}_m} (z_{\bar{i}_l} - z_{\bar{i}_m}) \prod_{j<k} (z_j - z_k)^2. \quad (11)$$

Here the indices j and k run over all particles, $\{i_m\}$ represent a subset of M indices, and $\{\bar{i}_m\}$ represent the remaining $N - M$ indices. While one can write general M-quasiparticle wave functions similar to the two particle case (6), it is only the maximum density droplet (8) that allows for a simple expression in terms of conformal blocks [17]. Using $[\varphi_1(z), \varphi_2(w)] = 0$ and

$\langle \varphi_i(z), \varphi_i(w) \rangle = -\ln(z - w)$, one can show that the V_2's, just as the V_1's but different from the P's, are fermionic operators, satisfying $V_2(z_i)V_2(z_j) + V_2(z_j)V_2(z_i) = 0$, and also that V_1 and V_2 commute. If we want to interpret V_2 as a composite *electron* operator, it should have the same charge as V_1. This is ensured if we redefine the charge density operator as $J(z) = (i/\sqrt{3})\partial \varphi_1(z) + (i/\sqrt{15})\partial \varphi_2(z)$, consistent with the field φ_2 being compactified with radius $R^2 = 15$.

Taking an equal number of V_1's and V_2's, i.e., $N = 2M$, charge neutrality implies $\nu = 1/3 + 1/15 = 2/5$. To see that (11) in this case *exactly* reproduces the CF wave function for the $\nu = 2/5$ state, we note that the latter is given by

$$\Psi_{Mqp} = \mathcal{P}_{\text{LLL}} \begin{vmatrix} \bar{z}_1 & \cdot & \cdot & \bar{z}_N \\ \bar{z}_1 z_1 & \cdot & \cdot & \bar{z}_N z_N \\ \cdot & \cdot & \cdot & \cdot \\ \bar{z}_1 z_1^{M-1} & \cdot & \cdot & \bar{z}_N z_N^{n-1} \\ 1 & \cdot & \cdot & 1 \\ z_1 & \cdot & \cdot & z_N \\ \cdot & \cdot & \cdot & \cdot \\ z_1^{M-1} & \cdot & \cdot & z_N^{M-1} \end{vmatrix} \prod_{j<k}(z_j - z_k)^2.$$

Using the Laplace expansion by the first M rows of the determinant, then carrying out the lowest LL projection (indicated by the symbol $\mathcal{P}_{\text{LLL}}$) by moving all of the $\bar{z}$'s to the left, and making the replacement $\bar{z}_j \rightarrow 2\partial_j$ reproduces (11), apart from a normalization factor. This proof trivially generalizes to $m = 2p + 1$, yielding the $\nu = 2/(4p + 1)$ states. Equation (8) can be interpreted as a "condensate" of quasiparticles, which resembles the approach of Halperin and Haldane [4,18]. Note, however, that it differs from a hierarchical wave function in that no explicit quasiparticle coordinates appear, and instead some of the electron coordinates themselves are altered to generate quasiparticles as explained above.

To create quasiholes in the $2/5$ state, the operator $H_{1/3}(\eta)$ given in (2) is no longer appropriate since it does not give holomorphic electron wave functions. That is accomplished by using either of

$$H_{2/5} = e^{(i/\sqrt{3})\varphi_1(\eta) + (i/\sqrt{15})\varphi_2(\eta)} \quad \text{or}$$
$$H_{1/5} = e^{(i/\sqrt{3})\varphi_1(\eta) - (2i/\sqrt{15})\varphi_2(\eta)}, \quad (12)$$

with charges $2/5$ and $1/5$, respectively. The explicit electron wave functions are obtained by inserting the operators (12) in the correlator (10).

By introducing more bosonic fields, the above construction can be generalized to wave functions for compact states of CFs involving still higher CF Landau levels. For example, the vertex operator for placing composite fermions in the third CF-LL is given by [9]

$$V_3(z) = \partial^2 e^{i(2/\sqrt{3})\varphi_1(z)} e^{i(2/\sqrt{15})\varphi_2(z)} e^{i(\sqrt{7/5})\varphi_3(z)}. \quad (13)$$

A general state in the Jain series $\nu = n/(2np + 1)$ requires n different fermionic operators V_s, $s = 1 \ldots n$, which are

constructed from n free, compact, boson fields φ_s and $n - 1$ derivatives. Again, the pertinent antisymmetrized CFT correlator exactly reproduces the corresponding CF wave functions. This also provides for a natural connection between the CF theory and Wen's general description of topological fluids based on Chern-Simons actions and, hence, candidate edge theories for the CF states. We have shown [9] that for the $\nu = 2/5$ and $\nu = 3/7$ states the effective theory based on the fields φ_i, $i = 1, 2, 3$, exactly reproduces Wen's results [10].

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