Nonautonomous Solitons in External Potentials

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Novel soliton solutions for the nonautonomous nonlinear Schrödinger equation models with linear and harmonic oscillator potentials substantially extend the concept of classical solitons and generalize it to the plethora of nonautonomous solitons that interact elastically and generally move with varying amplitudes, speeds, and spectra adapted both to the external potentials and to the dispersion and nonlinearity variations. The nonautonomous soliton concept can be applied to different physical systems, from hydrodynamics and plasma physics to nonlinear optics and matter waves, and offer many opportunities for further scientific studies.

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Zabusky and Kruskal introduced for the first time the soliton concept to characterize nonlinear solitary waves that do not disperse and preserve their identity during propagation and after a collision [1]. The Greek ending "on" is generally used to describe elementary particles, and this word was introduced to emphasize the most remarkable feature of these solitary waves. This means that the energy can propagate in the localized form and that the solitary waves emerge from the interaction completely preserved in form and speed with only a phase shift. Because of these defining features, the classical soliton is being considered as the ideal natural data bit. The optical soliton in fibers presents a beautiful example in which an abstract mathematical concept has produced a large impact on the real world of high technologies [2–4].

The classical soliton concept was developed for nonlinear and dispersive systems that have been autonomous; namely, time has only played the role of the independent variable and has not appeared explicitly in the nonlinear evolution equation. A not uncommon situation is one in which a system is subjected to some form of external time-dependent force. Such situations could include repeated stress testing of a soliton in nonuniform media with time-dependent density gradients; these situations are typical both for experiments with temporal or spatial optical solitons, soliton lasers, and ultrafast soliton switches and logic gates [2–4]. The formation of matter-wave solitons by magnetically tuning the interatomic interaction near a Feshbach resonance provides a good example of a nonautonomous system [5,6].

Historically, the study of soliton propagation through density gradients began with the pioneering work of Tappert and Zabusky [7]. As early as in 1976 Chen and Liu [8] substantially extended the concept of classical solitons to the accelerated motion of a soliton in a linearly inhomogeneous plasma. It was discovered that for the nonlinear Schrödinger equation (NLSE) model with a linear external potential the inverse scattering transform method (IST) [9] can be generalized by allowing the time-varying eigenvalue (TVE), and as a consequence of this, the solitons with time-varying velocities (but with time invariant amplitudes) have been predicted [8]. At the same time Calogero and Degasperis [10] introduced the general class of soliton solutions for the nonautonomous Korteweg–de Vries models with varying nonlinearity and dispersion. More recently, different aspects of one-soliton and multisoliton dynamics were investigated by Konotop *et al.* [11] for the discrete NLSE models. The "ideal" solitonlike interaction scenarios among nonautonomous solitons have been studied in [12,13] for the generalized NLSE models with varying dispersion, nonlinearity, and dissipation or gain.

Behaving like a particle, a soliton may be accelerated through a potential difference and reflected from the potential boundaries, and consequently, two general questions naturally arise: What happens with a "classical" soliton "beyond the autonomy" when the external potential is not only a function of coordinate but also a function of time? Do solitons still exist and maintain their identities through nonlinear interactions in time-dependent external fields?

We show that in the framework of the generalized nonautonomous NLSE model

$$i\frac{\partial Q}{\partial t} + \frac{D(t)}{2}\frac{\partial^2 Q}{\partial x^2} + \sigma R(t)|Q|^2 Q - 2\alpha(t)xQ - \frac{\Omega^2(t)}{2}x^2 Q = 0 \quad (1)$$

under the condition that dispersion $D(t)$, nonlinearity $R(t)$, and confining harmonic potential satisfy the exact integrability scenario

$$-\Omega^2(t)D(t) = \frac{d^2}{dt^2}\ln D(t) + R(t)\frac{d^2}{dt^2}\frac{1}{R(t)} - \frac{d}{dt}\ln D(t)\frac{d}{dt}\ln R(t), \quad (2)$$

the basic property of classical solitons, to interact elasti-

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cally, holds true, but the novel feature arises; namely, both amplitudes and speeds of the solitons, and consequently their spectra, during the propagation and after the interaction are no longer the same as those prior to the interaction. All nonautonomous solitons generally move with varying amplitudes $\eta(t)$ and speeds $\kappa(t)$ adapted to both the external potentials and the dispersion $D(t)$ and nonlinearity $R(t)$ changes. Equation (1) is written here in standard soliton units, as they are commonly known [2–6].

To show the exact integrability of Eq. (1) let us represent it as the condition for compatibility of a pair of linear differential equations:

$$\hat{\mathcal{F}}_t - \hat{\mathcal{G}}_x + [\hat{\mathcal{F}}, \hat{\mathcal{G}}] = 0. \tag{3}$$

This equation must be valid for all values of complex TVE $\Lambda(t) = \kappa(t) + i\eta(t)$ and is known as the generalization of Lax pair [14] defining the set of the Dirac-type eigenvalue equations for scattering potential $Q(x, t)$:

$$\psi_x = \hat{\mathcal{F}}\psi(x, t), \qquad \psi_t = \hat{\mathcal{G}}\psi(x, t). \tag{4}$$

Following the general strategy based on the IST-TVE method of Chen and Liu [8] we have constructed the desired matrices $\hat{\mathcal{F}}$ and $\hat{\mathcal{G}}$:

$$\hat{\mathcal{F}} = \begin{pmatrix} -i\Lambda(t) & \sqrt{\sigma}q(x,t) \\ -\sqrt{\sigma}q^*(x,t) & i\Lambda(t) \end{pmatrix}, \tag{5}$$

$$\hat{\mathcal{G}} = i\begin{pmatrix} \frac{\sigma}{2}D|q|^2 - \alpha x & \sqrt{\sigma}D(\frac{1}{2}q_x - i\Theta xq) \\ \sqrt{\sigma}D(\frac{1}{2}q_x^* + i\Theta xq^*) & -\frac{\sigma}{2}D|q|^2 + \alpha x \end{pmatrix} - i\Lambda D\begin{pmatrix} \Theta x & i\sqrt{\sigma}q \\ -i\sqrt{\sigma}q^* & -\Theta x \end{pmatrix} - i\Lambda^2 D\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

where for the sake of mathematical elegancy we made the reduction $q(x, t) = \sqrt{R(t)/D(t)}Q(x, t)\exp[ix^2\Theta(t)/2]$ with the self-induced soliton phase shift $\Theta(t) = -W[(R(t), D(t)]/D^2(t)/R(t)$ dependent on the Wronskian $W[(R(t), D(t)] = RD'_t - DR'_t$. It is straightforward to verify that the Lax Eq. (3) with matrices (5) and (6) provides the nonautonomous model (1) under condition (2) and the eigenvalue given by

$$\Lambda(t) = \frac{R(t)}{D(t)}\left[\Lambda(0) + \int_0^t \alpha(\tau)\frac{D(\tau)}{R(\tau)}d\tau\right], \tag{7}$$

where we assumed, without loss of generality, that $D(0) = R(0) = 1$.

Having obtained the eigenvalue equations for scattering potential Eq. (4) and keeping in mind the variety of physical applications of the model in hand, we can write down the general solutions for bright $Q^+(x, t \mid \sigma = +1)$ and dark $Q^-(x, t \mid \sigma = -1)$ nonautonomous solitons:

$$Q^+(x,t) = 2\eta(t)\sqrt{\frac{D(t)}{R(t)}}\operatorname{sech}[\xi(x,t)] \times e^{-i\{[\Theta(t)/2]x^2 + 2\kappa(t)x + 2\int_0^t D(\tau)[\kappa^2(\tau) - \eta^2(\tau)]d\tau\}}, \tag{8}$$

$$\xi(x, t) = 2\eta(t)x + 4\int_0^t D(\tau)\eta(\tau)\kappa(\tau)d\tau, \tag{9}$$

$$Q^-(x,t) = 2\eta(t)\sqrt{\frac{D(t)}{R(t)}}\left[\sqrt{(1-a^2)} + ia\tanh\zeta(x,t)\right] \times e^{-i\{[\Theta(t)/2]x^2 + 2K(t)x + 2\int_0^t D(\tau)[K^2(\tau) + 2\eta^2(\tau)]d\tau\}}, \tag{10}$$

$$\zeta(x, t) = 2a\eta(t)x + 4a\int_0^t D\eta\left[\eta\sqrt{(1-a^2)} + K\right]d\tau, \tag{11}$$

where $K(\tau) = R(\tau)/D(\tau)\int_0^\tau \alpha(\tau')D(\tau')/R(\tau')d\tau'$ and parameter a designates the depth of modulation (the blackness) of gray soliton and its velocity against the background. For optical applications, Eq. (10) can be easily transformed into the Hasegawa and Tappert form for dark soliton [2–4] under condition $\kappa_0 = \eta_0\sqrt{(1-a^2)}$. Notice that the solutions considered here hold only when the nonlinearity, dispersion, and confining harmonic potential are related by Eq. (2), and both $D(t) \neq 0$ and $R(t) \neq 0$ for all times by definition.

Example 1. Chirped nonautonomous optical solitons with moving spectra (colored solitons).—The transition to the problems of optical solitons is accomplished by the substitution $x \to T$ (or $x \to X$); $t \to Z$ and $Q^+(x, t \mid \sigma = +1) \to U^+(Z, T(\text{or } X))$ for bright solitons, and $[Q^-(x, t \mid \sigma = -1)]^* \to U^-(Z, T(\text{or } X))$ for dark solitons, where the asterisk denotes the complex conjugate, Z is the normalized distance, and T is the retarded time for temporal solitons, while X is the transverse coordinate for spatial solitons. Surprisingly, contrary to the well-studied model with linear potential [8], there exists a more general and exactly integrable nonautonomous model:

$$i\frac{\partial U}{\partial Z} + \frac{\sigma}{2}R(Z)e^{-\theta_0\int_0^Z R(z)dz}\frac{\partial^2 U}{\partial T^2} + R(Z)|U|^2U = 2\sigma\alpha(Z)TU. \tag{12}$$

Solitons with nontrivial self-induced phase shifts and varying amplitudes, speeds, and spectra for Eq. (12) are given in quadratures by Eqs. (8)–(11) under condition $\Omega^2(Z) = 0$. Hidden symmetry parameter θ_0 (initial linear phase chirp) in Eq. (12) separates the newly discovered class of solitons from the Chen and Liu model [8], where $D(t) = R(t) = 1$, $\alpha = \alpha_0 = \text{const}$, and $\sigma = +1$, $\theta_0 = 0$. It should be emphasized that the accelerated solitons predicted by Chen and Liu in plasma [8] were discovered in nonlinear fiber optics only a decade later [15]. Let us show that the so-called Raman colored solitons can be approximated by this equation. Self-induced Raman effect (also known as the soliton self-frequency shift) is being described by an additional term in the NLSE, namely, $-\sigma_R U \partial |U|^2/\partial T$ [2–4], where σ_R originates from the frequency dependent Raman gain [2–4]. Assuming that soliton amplitude does not vary significantly during self-scattering $|U|^2 = \eta^2 \mathrm{sech}^2(\eta T)$, we obtain that $\sigma_R \partial |U|^2/\partial T \approx -2\sigma_R \eta^4 T = 2\alpha_0 T$ and, as follows from Eq. (7), $dv/dZ = \sigma_R \eta^4/2$ where $v = \kappa/2$. The result of soliton perturbation theory [2–4] gives $dv/dZ = 8\sigma_R \eta^4/15$. This fact explains the remarkable stability of colored Raman solitons that is guaranteed by the property of the exact integrability of the Chen and Liu model [8]. The more general model of Eq. (12) and its exact soliton solutions open the possibility of designing an effective soliton compressor, for example, by drawing a fiber with $R(Z) = 1$ and $D(Z) = \exp(-\theta_0 Z)$ [16,17]. It seems very attractive to use the results of the nonautonomous solitons concept in ultrashort photonic applications and soliton lasers design [16,17]. Another interesting feature of the novel solitons is associated with the nontrivial dynamics of their spectra given by Eq. (7): if dispersion and nonlinearity evolve in unison, $D(t) = R(t)$ or $D = R = 1$, the solitons propagate with identical spectra but with totally different time-space behavior. We display the main remarkable features of nonautonomous colored solitons in Fig. 1.

Example 2. Nonautonomous spatial solitons in graded-index nonlinear waveguides.—Recently, Ponomarenko and Agrawal [18] have discovered a wide class of spatial solitonlike self-similar waves which can propagate in nonlinear graded-index amplifiers. They discovered the continuum radiation emitted by self-similar waves and nontrivial dynamics of the solitary wave center. These features do not exist for nonautonomous solitons, and the basic property of nonautonomous solitons—to preserve their identity during propagation and after a collision—holds true. As follows from Eqs. (1) and (2), the exactly integrable model governing of spatial solitons propagation in graded-index waveguide is reduced to

$$i\frac{\partial U}{\partial Z} + \frac{\sigma}{2} D \frac{\partial^2 U}{\partial X^2} + |U|^2 U + \frac{\sigma}{2D}\frac{d}{dZ}\left(\frac{1}{D}\frac{dD}{dZ}\right) X^2 U = 0. \tag{13}$$

Dispersion decreasing fibers (DDF) have found wide use in optical solitons compression techniques [2–4,16,17]. The integrability scenario Eq. (13) provides the repulsive parabolic potential $V(Z, X) = \sigma \beta^2 X^2/(1 + \beta Z)/2$ under the condition that $D(Z) = (1 + \beta Z)^{-1}$. We illustrate the main features of these nonautonomous spatial solitons in Fig. 2. When coherent (phase dependent) nonlinear short-range forces among solitons dominate [19], two soliton bound state can be formed despite the repulsive character of parabolic potential Fig. 2(b).

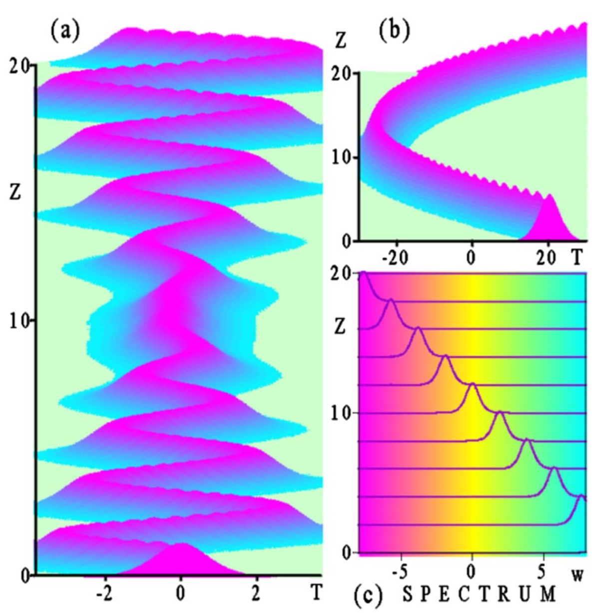

FIG. 1 (color online). Evolution of colored solitons and its spectra calculated within the framework of the generalized model given by Eq. (12) $\sigma = 1$ after choosing the soliton management parameters (a) $D(Z) = R(Z) = \cos(3Z)$ and (b) $D(Z) = R(Z) = 1$. In both cases, solitons propagate with identical spectra (c) and $\alpha(Z) = 1$ and $\Omega^2(Z) = 0$. Input conditions: $\eta_0 = 0.5$ and $\kappa_0 = 5.0$.

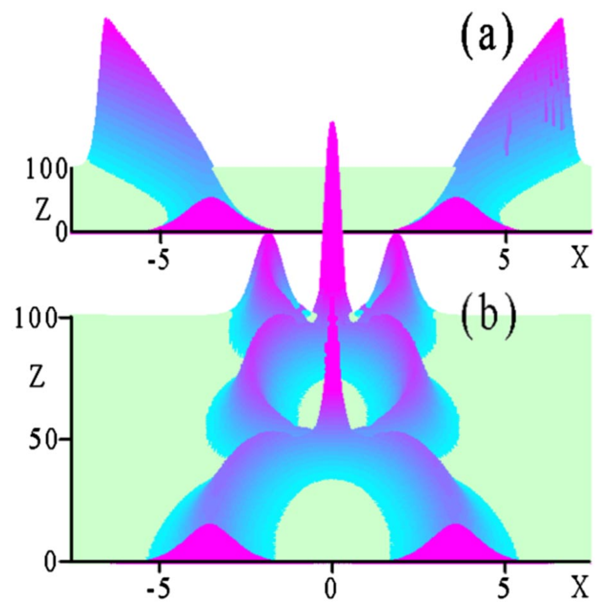

FIG. 2 (color online). Self-compression of spatial nonautonomous solitons and dynamics of their interaction calculated within the framework of the model Eq. (13) after choosing the soliton management parameters $D(Z) = 1/(1 + \beta Z)$ for (a) $\beta = 0.03$ and (b) $\beta = 0.01$. Input conditions: two identical soliton pulses $Q(T - \Delta T/2)$ and $Q(T + \Delta T/2)$ with equal amplitudes $\eta_0 = 0.5$ and speeds $\kappa_0 = 0$ initially separated in the time domain by $\Delta T = 7.0$.

Example 3. Matter-wave solitons management concept.—Equation (1) can be considered as the exactly integrable model for harmonically trapped one-dimensional (cigar-shaped [5,6]) Bose-Einstein condensate (BEC) which, in particular, case $D(t) \equiv 1$, can be reduced to

$$i\frac{\partial Q}{\partial t} + \frac{1}{2}\frac{\partial^2 Q}{\partial x^2} = \left[2\alpha x - \sigma R|Q|^2 - \frac{R}{2}\frac{d^2}{dt^2}\left(\frac{1}{R}\right)x^2\right]Q \quad (14)$$

with a plethora of nonautonomous soliton solutions given by Eqs. (8)–(11). A BEC is acted upon by gravity, like all matter, and because of this, an additional linear potential arises in Eq. (14). As follows from Eq. (14), variations of magnetically tuned the interatomic interaction strength $R(t)$ must be consistent with variations of the confining potential $\Omega^2(t) = R(t)[R^{-1}(t)]''_{tt}$ [19]. That means that near a Feshbach resonance when nonlinearity has a dispersive form $R(t) = R(0)/(1 - \gamma t)$, bright and dark matter-wave solitons can be stabilized even without a trapping potential [20]. In the case of periodically varying interaction strength among atoms, variations of confining harmonic potential are bound to have sign reversal to support the stable matter-wave solitons generation [20].

In conclusion, we have discovered and analytically described a novel class of soliton solutions for the nonautonomous NLSE models with linear and harmonic oscillator potentials, which substantially extend the concept of classical solitons and generalize it to the elastically interacting nonautonomous solitons moving with varying amplitudes, speeds, and spectra and adapted to both the external potentials and the dispersion and nonlinearity variations. We stress that these solitons represent exact, nonperturbative solutions for nonlinear evolution Eq. (1), and they are drastically different from the well-known results of soliton perturbation theory. Their remarkable properties have been proven in our computer simulations with the accuracy as high as 10^{-9}.

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