Gravitomagnetic Influence on Gyroscopes and on the Lunar Orbit

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Gravitomagnetism—a motional coupling of matter analogous to the Lorentz force in electromagnetism—has observable consequences for any scenario involving differing mass currents. Examples include gyroscopes located near a rotating massive body and the interaction of two orbiting bodies. In the former case, the resulting precession of the gyroscope is often called "frame dragging" and is the principal measurement sought by the Gravity Probe-B experiment. The latter case is realized in the Earth-Moon system, and the effect has in fact been confirmed via lunar laser ranging to approximately 0.1% accuracy—better than the anticipated accuracy of the Gravity-Probe-B result. This Letter shows the connection between these seemingly disparate phenomena by employing the same gravitomagnetic term in the equation of motion to obtain both gyroscopic precession and modification of the lunar orbit.

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Part of the post-Newtonian gravitational interaction between two mass elements, when both are in motion, has been called "gravitomagnetism," in analogy with the magnetic force between moving charges. The gravitomagnetic interaction is part of the more general $1/c^2$-order motional corrections to Newtonian gravity that result from field theories such as Einstein's general relativity and scalar-tensor generalizations [1,2]. A total package of velocity-dependent corrections is required so that the gravitational equation of motion remains consistent when expressed in different asymptotic inertial reference frames. If Lorentz invariance of local gravitational physics is imposed by empirical constraint, the package of motional corrections is additionally limited in structure.

The gravitomagnetic interaction of general relativity was first studied by Lense and Thirring in 1918, and it was shown to produce both accelerations of and torques on two neighboring rotating bodies. Others, viewing this phenomenon geometrically, have coined the interpretive name "inertial frame dragging" from rotating matter. It has also been shown that the gravitomagnetic interaction plays a part both in shaping the lunar orbit at a level (part in a thousand) readily observable by laser ranging [3] and in contributing to the periastron precession of binary and especially double pulsars [4].

For applications to the analysis of gravitational phenomena, a general metric tensor field expansion for the gravitational potentials in a broad class of theories was developed by Will and Nordtvedt [5,6]. This parametrized post-Newtonian (PPN) framework yields a gravitomagnetic contribution to the equation of motion, which in the Lorentz-invariant case is

$$a_i = (2 + 2\gamma)\sum_j \frac{\mu_j}{c^2 r_{ij}^3} v_i \times (v_j \times r_{ij}), \qquad (1)$$

where v_i and v_j are the velocities of bodies i and j in the chosen asymptotic inertial coordinate system [3]. The vector r_{ij} when combined with the fraction μ_j/r_{ij}^3 constitutes the Newtonian gravitational acceleration of mass i toward mass j. In geometric language, the PPN factor γ quantifies the amount of space curvature produced per unit mass. In general relativity, $\gamma = 1$. Metric theories allowing preferred inertial frame effects (absence of local Lorentz invariance) add the parameter $\alpha_1/4$ to the $(2 + 2\gamma)$ prefactor in Eq. (1), but lunar laser ranging as well as other solar system observations constrain α_1 to be less than 10^{-4} [7]. The conclusions of this Letter are not dependent on our choice to work in the PPN framework [8].

We can ask what effect the gravitomagnetic term of Eq. (1) has on a gyroscope outside of a rotating spherical mass. We define the gravitomagnetic field by

$$G_{ij} \equiv (2 + 2\gamma)\sum_j \frac{\mu_j}{c^2 r_{ij}^3}(v_j \times r_{ij}), \qquad (2)$$

so that $a_i = v_i \times G_{ij}$ in analogy to the electromagnetic Lorentz force. Considering a small gyroscope, the G_{ij} vector field is calculated at the gyroscope center and will be nearly constant across its body. To obtain the cumulative effect of mass elements moving within a body rotating at angular velocity Ω, the gravitomagnetic field is integrated over all mass elements j, each with $d\mu_j = G\rho(r_j)d^3 r_j$, where G is Newton's gravitational constant, and $\rho(r_j)$ is the mass density at radius r_j from the body center. Adopting a spherical coordinate system aligned with the rotation axis of the body, we describe the Cartesian vector $r_j = r\sin\theta\cos\phi\mathbf{i} + r\sin\theta\sin\phi\mathbf{j} + r\cos\theta\mathbf{k}$, and the vector to the gyroscope (placed in the $\phi = 0$ plane) is $r_i = a\sin\psi\mathbf{i} + a\cos\psi\mathbf{k}$, so that $r_{ij}^2 = a^2 + r^2 - 2ar(\sin\psi\sin\theta\cos\phi + \cos\psi\cos\theta)$. The velocity of mass element j is $v_j = \Omega r\sin\theta(-\sin\phi\mathbf{i} + \cos\phi\mathbf{j})$, so that

$$dG_{ij} = (2 + 2\gamma)\frac{\Omega r \sin\theta d\mu_j}{c^2 r_{ij}^3}\{(r\cos\theta - a\cos\psi)$$
$$\times (\cos\phi\mathbf{i} + \sin\phi\mathbf{j}) + (a\sin\psi\cos\phi - r\sin\theta)\mathbf{k}\}.$$
$$(3)$$

For the special case of a gyroscope situated over the pole for simplified integration ($\psi = 0$), and recognizing that the **k** component of the G_{ij} vector will be the only one to yield a nonzero angular integral, we find that

$$G_{ij}(\psi = 0) = -\frac{(2 + 2\gamma)G\Omega\mathbf{k}}{c^2}$$
$$\times \int_0^{2\pi} d\phi \int_{-1}^1 du \int_0^R dr \frac{\rho(r)r^4(1 - u^2)}{(a^2 + r^2 - 2aru)^{3/2}}$$
$$= -\frac{8\pi(2 + 2\gamma)G\Omega\mathbf{k}}{3c^2 a^3}\int_0^R dr\rho(r)r^4. \quad (4)$$

Here we used the identity $u = \cos\theta$ and note that the integral over u eliminates the r dependence in the denominator. Recognizing that the moment of inertia of a spherical body is

$$I = \frac{8\pi}{3}\int_0^R \rho(r)r^4 dr, \quad (5)$$

we see that

$$G_{ij}(\psi = 0) = -\frac{(2 + 2\gamma)GI_s\Omega}{c^2 a^3}\mathbf{k}, \quad (6)$$

where the s subscript represents the massive rotating body.

Orienting the gyroscope so that its spin axis is along the **i** direction, $\boldsymbol{\omega} = \omega\mathbf{i}$, the velocity of an element within the gyroscope is $\boldsymbol{v}_i = \boldsymbol{\omega} \times \boldsymbol{r}_i = -\omega r\cos\theta\mathbf{j} + \omega r\sin\theta\sin\phi\mathbf{k}$, where $\boldsymbol{r}_i$ is the vector position of a mass element within the gyroscope with respect to its center. Now all pieces are at hand to evaluate the acceleration of each mass element within the gyroscope due to the gravitomagnetic term. The force on each element is $d\boldsymbol{F}_i = dm_i\boldsymbol{a}_i = dm_i\boldsymbol{v}_i \times \boldsymbol{G}_{ij}$, and the torque on the gyroscope from this element is then $d\boldsymbol{\tau} = \boldsymbol{r}_i \times d\boldsymbol{F}_i$. Combining these steps,

$$d\boldsymbol{\tau} = \frac{(2 + 2\gamma)GI_s\Omega\omega r^2 \cos\theta dm_i}{c^2 a^3}(\cos\theta\mathbf{j} - \sin\theta\sin\phi\mathbf{k}). \quad (7)$$

Integrating this torque over the volume of the gyroscope, the **k** component integrates to zero in the ϕ integral, yielding

$$\boldsymbol{\tau} = \frac{(2 + 2\gamma)GI_s\Omega\omega}{c^2 a^3}\mathbf{j}\int_0^{2\pi}d\phi\int_{-1}^1 u^2 du\int_0^R\rho(r)r^4 dr$$
$$= \frac{(1 + \gamma)GI_s\Omega I_g\omega}{c^2 a^3}\mathbf{j}, \quad (8)$$

where the integral is seen to be one-half the rotational inertia of the gyroscope, which is denoted with a

subscript g. This torque will change the angular momentum vector of the gyroscope $\boldsymbol{L}_g = I_g\boldsymbol{\omega}$, such that the angle of the axis Φ precesses at a rate of $\dot{\Phi} = |\boldsymbol{\tau}|/|\boldsymbol{L}_g|$, so that

$$\dot{\Phi} = \frac{(1 + \gamma)GI_s\Omega}{c^2 a^3}. \quad (9)$$

The direction of precession indicated by Eq. (8) is one that takes the angular momentum—originally, in the **i** direction only—toward the **j** direction, meaning that the precession has the same sense as the rotation of the massive body. Had we developed an expression for $\boldsymbol{G}_{ij}(\psi = \frac{\pi}{2})$ at the equator of the rotating body, we would have found half the magnitude of the polar case and in the opposite direction. In general, the field

$$\boldsymbol{G}_{ij} = -\frac{(1 + \gamma)GI_s\Omega}{c^2 a^3}[3(\mathbf{k} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \mathbf{k}], \quad (10)$$

where $\hat{\boldsymbol{r}}$ is the unit radial vector. In a circular polar orbit with uniform angular rate in ψ, the gravitomagnetic field averages to

$$\langle\boldsymbol{G}_{ij}\rangle = -\frac{(1 + \gamma)GI_s\Omega}{2c^2 a^3}\mathbf{k}, \quad (11)$$

so that the net field shares the same direction as that over the pole, and therefore the net precession will be in the same sense as the rotating mass, but at one-fourth the polar rate. Summarizing,

$$\dot{\Phi}_{\text{polar orbit}} = \frac{(1 + \gamma)GI_s\Omega}{4c^2 a^3}. \quad (12)$$

Putting this in another form, where we reduce the rotational inertia to $I_s = fM_sR^2$, where $f = 0.33$ for the Earth [9], we have the more convenient form:

$$\dot{\Phi}_{\text{polar orbit}} = \frac{(1 + \gamma)f}{4}\frac{GM_s}{Rc^2}\left(\frac{R}{a}\right)^3\Omega. \quad (13)$$

For Gravity Probe-B (GP-B), in a 640 km altitude polar orbit, Eq. (13) yields 0.042 arcseconds per year, matching the mission expectation [10,11]. GP-B anticipates measuring this precession to $<1\%$ accuracy and perhaps as well as 0.1%–0.3%. Note that the gyroscope *spin* was not treated as an intrinsic property in deriving the gyroscope precession. Rather, the effect results from the integrated mass currents of mass elements in rotational motion.

In obtaining the effect of the gravitomagnetic term [Eq. (1)] on the lunar orbit relative to Earth, we treat the gravitomagnetic acceleration as a perturbation about an otherwise circular orbit. We start with an orbit obeying the two-body equation of motion:

$$\ddot{r} = -\frac{GM}{r^2} + r\omega^2 = a(r) + \frac{l^2}{r^3}, \quad (14)$$

where $a(r)$ is the central acceleration, and $l = r^2\omega$ is the (specific) angular momentum. We idealize the unperturbed orbit to have zero eccentricity and zero inclination, so that

the end result is accurate for the Moon at the 10% level or better. The deviation δr, resulting from a periodic acceleration perturbation $\vec{\delta a}$, then obeys

$$\ddot{\delta r} + \omega_0^2 \delta r = \delta a_r + 2\omega_0 \int^t \delta a_\tau dt', \qquad (15)$$

where ω_0 is the natural frequency for orbital perturbations, with $\omega_0^2 \approx 3\omega^2 - \frac{da}{dr} \approx \omega^2$. The acceleration $\vec{\delta a}$ has been decomposed into radial and tangential components, and t' is a time variable.

Expressing the triple cross-product in Eq. (1) as the equivalent dot-product relationship, we find that the gravitomagnetic acceleration of the Moon is

$$\boldsymbol{a}_m \equiv \vec{\delta a} = \frac{\mu_e(2 + 2\gamma)}{c^2 a^2}[\hat{\boldsymbol{r}}(\boldsymbol{v}_m \cdot \boldsymbol{v}_e) - \boldsymbol{v}_e(\boldsymbol{v}_m \cdot \hat{\boldsymbol{r}})], \quad (16)$$

where $\hat{\boldsymbol{r}}$ is the unit vector from the Earth to the Moon, and a is the Earth-Moon distance. Equation (16) is rewritten as

$$\vec{\delta a} = \frac{(2 + 2\gamma)GM}{c^2 a^2}[\hat{\boldsymbol{r}}(V^2 + \boldsymbol{V} \cdot \boldsymbol{u}) - \boldsymbol{V}(\boldsymbol{V} \cdot \hat{\boldsymbol{r}} + \boldsymbol{u} \cdot \hat{\boldsymbol{r}})],$$
$$(17)$$

with the Earth's velocity around the Sun being $\boldsymbol{V}$ and the Moon's velocity being $\boldsymbol{V} + \boldsymbol{u}$, where u is approximately 30 times smaller than V in magnitude.

Note that $\boldsymbol{u}$ represents to sufficient accuracy the geocentrically viewed orbital velocity of the Moon around the Earth. Thus, under the assumption of a circular orbit (about which we examine the perturbation), $\boldsymbol{u} \cdot \hat{\boldsymbol{r}} = 0$. Likewise, if we define $\hat{\boldsymbol{\tau}}$ to be the tangential orbit vector at the Moon that is perpendicular to $\hat{\boldsymbol{r}}$, $\boldsymbol{u} \cdot \hat{\boldsymbol{\tau}} = u$. Under the assumption that the Earth is in a circular orbit about the Sun, the relationship between $\boldsymbol{V}$ (perpendicular to Earth-Sun line) and $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\tau}}$ picks out the synodic phase angle of the Moon, D. Specifically, $\boldsymbol{V} \cdot \hat{\boldsymbol{r}} = -V \sin D$ and $\boldsymbol{V} \cdot \hat{\boldsymbol{\tau}} = -V \cos D$. Similarly, $\boldsymbol{V} \cdot \boldsymbol{u} = -Vu \cos D$. Leaving off the prefactor from Eq. (17) for now, and dealing only with the vector math, the radial component is

$$\delta a_r \propto V^2 - Vu \cos D - (-V \sin D)^2$$
$$= \tfrac{1}{2}V^2 + \tfrac{1}{2}V^2 \cos 2D - Vu \cos D, \qquad (18)$$

and the tangential component is

$$\delta a_\tau \propto -(-V \cos D)(-V \sin D) = -\tfrac{1}{2}V^2 \sin 2D. \quad (19)$$

The periodic accelerations consist of two categories: V^2 terms that have $2D$ angular dependence and Vu terms that have D angular dependence. We can treat each separately in solving Eq. (15). There is also a constant term in the expression for δa_r. We can ignore the constant term since it acts only to rescale the orbit in a nonperiodic way. We will deal first with the $2D$ terms, then look at the D terms.

First, we integrate the δa_τ term. Noting that the rate at which D advances is $\dot{D} = \omega - \Omega$, the difference between the lunar orbital frequency and the Earth's orbital fre-

quency, we can construct an arbitrary $2D$ argument as $[2(\omega - \Omega)t' + \phi]$, where ϕ is an arbitrary phase depending on the choice of $t' = 0$. The integral (without the numerical prefactor) is then

$$2\omega \int^t \sin[2(\omega - \Omega)t' + \phi]dt' = -\frac{\omega}{\omega - \Omega}\cos 2D + \text{const.}$$
$$(20)$$

We have consolidated any initial phase in the integration constant, effectively defining t so that $D = (\omega - \Omega)t$. As above, we can ignore the constant term in our periodic analysis. The differential equation becomes

$$\ddot{\delta r} + \omega^2 \delta r = \frac{(1 + \gamma)GM}{a^2 c^2} V^2\left[\cos 2D + \frac{\omega}{\omega - \Omega}\cos 2D\right]$$
$$= \frac{(1 + \gamma)GM}{a^2}\frac{V^2}{c^2}\frac{2\omega - \Omega}{\omega - \Omega}\cos 2D. \qquad (21)$$

The solution, in meters, is then

$$\delta r \approx -(1 + \gamma)\frac{V^2}{c^2}\frac{2 - \eta}{1 - \eta}\frac{a}{3 - 8\eta}\cos 2D$$
$$\approx -6.5\cos 2D \text{ m,} \qquad (22)$$

where we have made use of Kepler's relation ($\omega^2 a^3 = GM$) and define $\eta \equiv \Omega/\omega$, ignoring terms to second order in η.

The term proportional to Vu has no tangential part, so we immediately write the differential equation as

$$\ddot{\delta r} + \omega^2 \delta r = -\frac{(2 + 2\gamma)GM}{c^2 a^2}Vu \cos D, \qquad (23)$$

for which the solution is

$$\delta r = -\frac{(2 + 2\gamma)GM}{c^2 a^2}\frac{Vu}{\omega^2 - (\omega - \Omega)^2}\cos D$$
$$\approx -(1 + \gamma)\frac{Vu}{c^2}\frac{\omega}{\Omega}a \cos D \approx -3.4\cos D \text{ m.} \quad (24)$$

But a feedback process produced by the interaction of synodic perturbations and the $\cos 2D$ solar tidal distortion of the lunar orbit results in an amplification of $\cos D$ terms by the factor [12]

$$Q_{\text{res}} \approx \frac{1 - 2\eta}{1 - 7\eta} \approx 1.79, \qquad (25)$$

so that the corrected range oscillation is

$$\delta r \approx -(1 + \gamma)\frac{Vu}{c^2}\frac{\omega}{\Omega}\frac{1 - 2\eta}{1 - 7\eta}a \cos D \approx -6.1\cos D \text{ m.}$$
$$(26)$$

In summary, the gravitomagnetic perturbations of the lunar orbit are

$$\delta r_{2D} = -6.5 \cos 2D \text{ meters,}$$
$$\delta r_D = -6.1 \cos D \text{ meters.}$$

(27)

Lunar laser ranging (LLR) has been used for decades to provide a number of the most precise tests of general relativity, including tests of the weak and strong equivalence principles, the time rate of change of Newton's gravitational constant G, and geodetic precession, among others [13]. Equivalence principle violations would produce a $\cos D$ signal [14], though no $\cos 2D$ signal. Current fits to the archive of LLR data limit any net deviation of the $\cos D$ term in the lunar orbit to less than ≈ 4 mm from the orbit prescribed by general relativity [13]. Likewise, the $\cos 2D$ term is constrained at roughly the 8 mm level. Thus, barring a chance simultaneous violation of the equivalence principle *and* gravitomagnetism, the 4 mm constraint translates to a check on the ~ 6 m gravitomagnetism amplitude to better than 0.1% accuracy. Allowing for such a conspiracy, we must use the 8 mm $\cos 2D$ constraint (which is not influenced by equivalence principle violation) to establish a $\approx 0.15\%$ verification of the gravitomagnetic phenomenon. At this time, LLR provides the most precise test of this phenomenon—far better than the laser geodynamics satellites tests of the Lense-Thirring effect [15] and tests from binary pulsars [4]. This result is also likely better than the ultimate result from the GP-B experiment [10].

A new effort in LLR is poised to deliver order-of-magnitude improvements in range precision [16], which will translate into tighter constraints on the $\cos D$ and $\cos 2D$ amplitudes in the lunar orbit. Because these are periodic effects, their accurate determination requires only about a year of new data collection. Thus, a significantly improved test of this phenomenon is not far away.

Whether the mass elements in a body are moving commonly—as for the Earth and the Moon in orbital motion in the solar system—or as mass currents in the rotational manner of the spinning Earth and the gyroscope in GP-B, the total interaction between bodies must be dominated by a linear-order integration over the bodies' mass elements in both situations. Breaking weak-field superposition would be a radical and ultimately nonviable modification to gravity theory. If this linear-order gravitomagnetic interaction from PPN metric gravity [Eq. (1)] is altered in order to fit any anomalous GP-B observation, then either the $\cos D$ or $\cos 2D$ amplitudes, or both, in the Earth-Moon range as measured by LLR to half-centimeter accuracy will show anomalies under this new modeling—establishing a profound empirical clash. An added likely consequence of modifying gravitomagnetism would be destroying the total fit to the binary pulsar 1913+16 data, which includes a better than 1% match to general relativity's predicted gravitational radiation-reaction accelerations in that sys-

tem. Existing and robust observations already encumber the gravitomagnetic interaction.

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[8] Although the present analysis uses the equation of motion from the PPN formulation of long range, metric gravity, the conclusions we draw are not constrained by this choice. The phenomenology we explore—the $\boldsymbol{v}_i \times (\boldsymbol{v}_j \times \boldsymbol{g}_{ij})$ acceleration of Eq. (1)—generically results from modified metric field expansions or even nonmetric models of gravity at the $1/c^2$ post-Newtonian level. If such an interaction is used to explain precessional effects in a gyroscope experiment, then it will also be present to perturb the lunar orbit and binary pulsar orbits, regardless of its parametrized strength in any particular model. Any attempt to suppress the gravitomagnetic influence on the lunar orbit relative to that on a low-orbiting gyroscope by a Yukawa-like modification to $\boldsymbol{g}_{ij}$—whether metric or nonmetric—would clash with strong constraints on the inverse-square nature of $\boldsymbol{g}_{ij}$ determined via satellite and lunar laser ranging. Equation (1) is, of course, dependent on the asymptotic inertial frame in which analysis is performed, just as are magnetic forces within an electromagnetic system. Consistent formalisms can be used in any choice of frame to calculate observables; this permits using the convenient solar system barycentric frame for our analysis.

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