# Phase Diagram of a Strongly Interacting Polarized Fermi Gas in One Dimension

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Based on the integrable Gaudin model and local density approximation, we discuss the ground state of a one-dimensional trapped Fermi gas with imbalanced spin population, for an arbitrary attractive interaction. A phase separation state, with a polarized superfluid core immersed in an unpolarized superfluid shell, emerges below a critical spin polarization. Above it, coexistence of polarized superfluid matter and a fully polarized normal gas is favored. These two exotic states could be realized experimentally in highly elongated atomic traps, and diagnosed by measuring the lowest density compressional mode. We identify the polarized superfluid as having an Fulde-Ferrell-Larkin-Ovchinnikov structure, and predict the resulting mode frequency as a function of the spin polarization.

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Strongly attractive Fermi gases with imbalanced spin components are common in different branches of physics [1–8], so spin-polarized ultracold atomic gases are an atomic analog of many other exotic forms of matter. The true ground state of an attractive polarized Fermi gas remains elusive, because the standard Bardeen-Cooper-Schrieffer (BCS) mechanism requires the pairing of two fermions with opposite spin. Polarized Fermi gases cannot be explained by the BCS theory, due to mismatched Fermi surfaces. Various exotic forms of pairing have been proposed, including the deformed Fermi surface [3], interior gap pairing [4] or Sarma superfluidity [5], phase separation [6], and the inhomogeneous Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [7].

Recent measurements on polarized $^6$Li gases [1,2] near a Feshbach resonance provide a route towards testing these theories in experiment. However, the presence of a harmonic trap in these experiments makes it difficult to identify which if any of the different pairing schemes occurs. Theoretically, it is desirable to have an exactly soluble mode of polarized uniform Fermi gases to identify various pairing schemes, and clarify the issue of the trap.

In the present Letter we report on the exact ground state of a homogeneous 1D polarized Fermi gas [9–11] with attractive intercomponent interactions at zero temperature. We then study the phase diagram of an inhomogeneous Fermi gas under harmonic confinement, using the local density approximation (LDA). We complement this with a mean-field Bogoliubov–de Gennes (BdG) theory in the weak-coupling limit, where the phase fluctuations are small, in order to clarify the physical meaning of the solutions. Collective mode frequencies are also calculated as an experimental diagnostic, thus extending previous work on the unpolarized case [12–14].

As well as being a theoretical test bed for the ground-state issue, 1D Fermi gases in traps can be realized using two-dimensional optical lattices, where the radial motion of atoms is frozen out. Thus, one can experimentally check these quantum many-body predictions, which has also been recently carried out for the 1D Bose gas [15].

The following remarkable features are found: (i) In the ground state of a uniform system, we find three distinct phases with increasing chemical potential difference between species: an unpolarized BCS superfluid, a polarized superfluid, and a fully polarized normal state. The polarized superfluid is most widespread, and reduces to the FFLO-type for weak interactions. Therefore, it is relatively easy to observe the FFLO physics in one dimension, as anticipated in previous approximate studies [16]. In earlier work the phase diagram was not conclusive, as the nature of the transition from BCS to FFLO states was under debate [16]. (ii) Within the LDA, we consider the inhomogeneous phase diagram of the trapped gas. This leads to a *phase separation*, as the inhomogeneous cloud separates into either a mixture of a polarized superfluid and an unpolarized superfluid, or a polarized superfluid and a fully polarized normal gas. (iii) We calculate the longitudinal size of the two spin components and the frequency of the lowest density compressional mode. These quantities give a measurable fingerprint of the whole phase diagram.

We describe a 1D polarized Fermi gas with $N = N_\uparrow + N_\downarrow$ fermions each of mass $m$ and spin polarization $P = (N_\uparrow - N_\downarrow)/N > 0$ in a harmonic trap, by

$$\mathcal{H} = \mathcal{H}_0 + \sum_{i=1}^{N} \frac{1}{2} m\omega_{\mathrm{ho}}^2 x_i^2, \tag{1}$$

where

$$\mathcal{H}_0 = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + g_{1\mathrm{D}} \sum_{i=1}^{N_\uparrow} \sum_{j=1}^{N_\downarrow} \delta(x_i - x_j) \tag{2}$$

is the Hamiltonian of Gaudin model of a spin 1/2 Fermi gas attracting via a short range potential $g_{1\mathrm{D}}\delta(x)$ [9]. The

 

coupling constant $g_{1D}$ ($<0$) can be expressed through the 1D scattering length $a_{1D}$, $g_{1D} = -2\hbar^2/(ma_{1D})$. A two-fermion bound state arises once $N_\downarrow > 0$, with binding energy $\epsilon_b = \hbar^2/(ma_{1D}^2)$.

In the absence of the harmonic trap, the integrable Gaudin model, Eq. [9], can be solved exactly using Bethe's ansatz [9,10]. Introducing linear number densities, $n = N/L$ and $n_\sigma = N_\sigma/L$ ($\sigma = \uparrow, \downarrow$), where $L$ is the size of the system, the uniform gas is characterized by the polarization $p = (n_\uparrow - n_\downarrow)/n > 0$ and a dimensionless parameter $\gamma = -mg_{1D}/(\hbar^2 n) = 2/(na_{1D}) > 0$. The ground state is obtained by numerically solving the Gaudin integral equations [9,10]. We have clarified the physical nature of the resulting solutions by also solving the weak-coupling mean-field BdG equations.

Figure 1 shows the energy per particle, $E_{\rm hom}/N$, the chemical potential of spin up fermions, $\mu_{{\rm hom},\uparrow} = \partial E_{\rm hom}/\partial N_\uparrow$, and the chemical potential difference, $\delta\mu_{\rm hom} = \partial E_{\rm hom}/\partial(N_\uparrow - N_\downarrow)$, as a function of polarization $p$ at two intermediate interaction strengths $\gamma = 0.5$ and $\gamma = 2.0$. The mean-field calculations lead to the same general behavior. Their asymptotic behavior in the weak and strong coupling limits may be understood analytically. Of particular interest is the chemical potential difference. In the weakly interacting limit of $\gamma \ll 1$, $\delta\mu_{\rm hom}$ is given by ($\gamma \ll \max\{p, 1-p\}$)

$$\delta\mu_{\rm hom} = \frac{\hbar^2 n^2}{2m}\frac{\pi^2}{2}p + \frac{\hbar^2 n^2}{2m}\gamma p + \cdots, \qquad (3)$$

where the first term on the right-hand side is the Fermi energy difference of an ideal polarized gas, and the second term arises from the mean-field Hartree-Fock interactions. The chemical potential difference increases with increasing $\gamma$, and reaches a half of the binding energy of bound states in the strongly attracting limit of $\gamma \to \infty$,

$$\delta\mu_{\rm hom} = \frac{\epsilon_b}{2} - \frac{\hbar^2 n^2}{2m}\frac{\pi^2(1-p)^2}{16} + \frac{\hbar^2 n^2}{2m}\pi^2 p^2 + \cdots. \qquad (4)$$

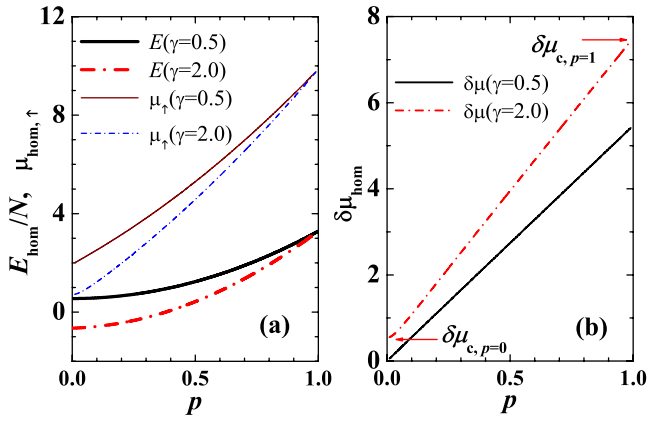


FIG. 1 (color online). The energy per particle, chemical potential for spin up atoms (a), and chemical potential difference (b) in units of $\hbar^2 n^2/2m$ as a function of the spin polarization at two coupling parameters $\gamma = 0.5$ and $\gamma = 2.0$.

The first two terms in magnitude coincide with the chemical potential of a Tonks-Girardeau bosonic gas of paired $N_\downarrow$ dimers [12,13], which is fermionized due to strong attractions, while the third term is equal to the chemical potential of residual unpaired $N_\uparrow - N_\downarrow$ noninteracting fermions. Therefore, in the strong coupling regime the system behaves like a coherent mixture of a molecular Bose gas and fully polarized single-species Fermi sea.

We analyze the phase structure of the ground state of uniform polarized Fermi gases (Fig. 2). Given an interaction strength the chemical potential difference at finite polarization $p$ is bounded by two critical values, $\delta\mu_{c,p=0}$ and $\delta\mu_{c,p=1}$, as indicated by arrows in Fig. 1(b) for $\gamma = 2.0$. Below $\delta\mu_{c,p=0}$, the gas remains in the BCS-like superfluid state with zero polarization (SF), while above $\delta\mu_{c,p=1}$, a fully polarized normal state is favored ($N$). In between, a superfluid state with finite polarization ($SF_P$) dominates. The mean-field calculation of a uniform gas shows that the $SF_P$ is of FFLO character, as the exactly soluble ground-state energy corresponds precisely with the FFLO solution in the weak-coupling limit. Physically $\delta\mu_{c,p=0}$ is the energy cost required to break pairs in unpolarized superfluid, i.e., the spin gap [13], while $\delta\mu_{c,p=1}$ is also associated with the pair-breaking (for the last pair), but is promoted upwards due to the Pauli repulsion from existing fermions. The dependence of $\delta\mu_{c,p=0}$ and $\delta\mu_{c,p=1}$ on the parameter $\gamma$ is reported in Fig. 2, resulting in a homogeneous phase diagram.

In units of $\hbar^2 n^2/2m$, these have the following limiting behavior: $\delta\mu_{c,p=0} \simeq 2\sqrt{\pi\gamma}\exp[-\pi^2/2\gamma]$ and $\delta\mu_{c,p=1} \simeq (\pi^2/2)$ as $\gamma \to 0$, while $\delta\mu_{c,p=0} \simeq \epsilon_b/2 + \pi^2/16$ and $\delta\mu_{c,p=1} \simeq \epsilon_b/2 + \pi^2$ as $\gamma \to \infty$. Both critical chemical potential differences saturate to the half of the binding energy in the strong coupling limit. Converting $\gamma$ into the chemical potential $\mu_{\rm hom} = \partial E_{\rm hom}/\partial N$, we can obtain the phase diagram in the $\mu_{\rm hom}$-$\delta\mu_{\rm hom}$ plane.

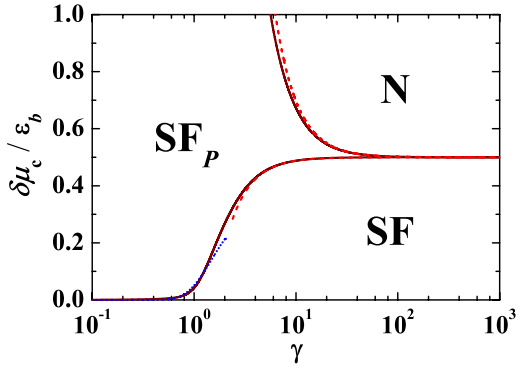


FIG. 2 (color online). Uniform phase diagram, displaying "normal" (N), unpolarized superfluid (SF), and polarized superfluid states ($SF_P$). At weak coupling or high density, the predominant $SF_P$ phase corresponds to the mean-field FFLO solution. The dashed and dotted lines are asymptotic behavior as described in the text.

We now turn to describe the 1D polarized gas in a harmonic trap. We partition the system into cells that can be treated locally as being uniform, with a local chemical potential. This LDA is applicable provided that the number of fermions in a cell is much greater than unitary, and the variation of the trap potential across the cell is small compared with the local Fermi energy and hence the interface effects are negligible [17]. Overall it requires a sufficient large local density, which implying $N \gg 1$, a condition readily satisfied in the 1D experiment. Note that the breakdown of LDA has been observed in the elongated 3D trap [17], when the linear density in the transverse axis becomes small. The LDA amounts to determining the chemical potential $\mu = (\mu_\uparrow + \mu_\downarrow)/2$ and the chemical potential difference $\delta\mu = (\mu_\uparrow - \mu_\downarrow)/2$ of the inhomogeneous gas from the local equilibrium condition,

$$\mu_{\mathrm{hom},\sigma}[n(x), p(x)] + \frac{1}{2} m\omega_{\mathrm{ho}}^2 x^2 = \mu_\sigma. \tag{5}$$

The normalization conditions are $N = \int n(x)dx$ and $NP = \int n(x)p(x)dx$, where $n(x)$ is the total linear number density and $p(x)$ the local spin polarization. By rescaling the chemical potentials, coordinate and linear density into dimensionless form [$\tilde{\mu}_\sigma = \mu_\sigma/\epsilon_b$, $\tilde{x} = x/(a_{\mathrm{ho}}^2/a_{1\mathrm{D}})$ and $\tilde{n} = n a_{1\mathrm{D}}$], the normalization equations can be rewritten as $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 = \int \tilde{n}(\tilde{x})d\tilde{x}$ and $(Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2)P = \int \tilde{n}(\tilde{x})p(\tilde{x})d\tilde{x}$. These expressions emphasize that the dimensionless coupling constant in a trap is controlled by $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2$, where $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 \gg 1$ corresponds to weak coupling while $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 \ll 1$ corresponds to the strongly interacting regime.

The qualitative feature of density profiles $n_\sigma(x)$ is simple to understand. Within the LDA, the local chemical potential $\mu(x)$ decreases parabolically away from the center of the trap while the local chemical potential difference $\delta\mu(x)$ stays constant. It is then clear from Fig. 2 that with a nonzero $P$ we always have a polarized superfluid in the center where the local chemical potential (interaction parameter) is large (small). Away from the center with decreasing local chemical potential, the Fermi gas goes into either an unpolarized superfluid ($\delta\mu < \epsilon_b/2$) or a fully polarized normal cloud ($\delta\mu > \epsilon_b/2$). Thus, there is a critical chemical potential difference $\delta\mu_c \equiv \epsilon_b/2$ that separates the inhomogeneous system into two phase separation states: a mixture of a polarized superfluid core and an unpolarized superfluid shell ($\mathrm{SF}_P$-SF), or a coexistence of a polarized superfluid at the center and a fully polarized normal gas outside ($\mathrm{SF}_P$-$N$).

The former phase is exotic, as the BCS-like superfluid state occurs at the edge of the trap, in marked contrast to the 3D case [1]. This is caused by the peculiar effect of low dimensionality, for which the gas becomes more nonideal with *decreasing* 1D density towards the edge of the trap, and hence the energy required to break the pairs approaches $\epsilon_b/2$ from below. As $\delta\mu < \epsilon_b/2$, there should be a fully paired region once the local critical chemical potential $\delta\mu_{c,p=0} > \delta\mu$, i.e., the BCS-like superfluid.

We show the density profile of each component in Fig. 3 with a typical coupling parameter $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 = 1$. In addition, we have performed a BdG calculation for the trapped gas and observe a FFLO-N phase. The resulting density profiles on the weak-coupling side are in perfect agreement with the LDA calculation, indicating unambiguously that the $\mathrm{SF}_P$ phase is a FFLO state.

We determine the phase diagram of the inhomogeneous polarized 1D Fermi gas as a function of the interaction strength and spin polarization by calculating the critical polarization $P_c$ that corresponds to $\delta\mu_c$ as a function of the coupling constant, and plot it in Fig. 4(a). The asymptotic behavior of $P_c$ can be computed analytically in the weak and strong coupling limits. We find that $P_c \simeq 1/(Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2)$ as $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 \to \infty$, and $P_c \simeq 1/5 - (256/225\pi^2)(0.4Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2)^{1/2}$ as $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 \to 0$.

We consider the experimental relevance of the two phase separation states, by calculating the size of the cloud and the lowest density compressional mode. These are readily detectable via absorption imaging. Figure 4(b) reports the evolution of the Thomas-Fermi radius of two spin components as a function of polarization at three different interaction couplings. The radius for spin up and down fermions is the same in the $\mathrm{SF}_P$-SF phase, but diverges in opposite directions in the $\mathrm{SF}_P$-$N$ phase.

Using a sum-rule approach, the frequency $\omega$ of the lowest density and spin density compressional (breathing) modes of 1D trapped gases can be calculated from the identity $\hbar^2\omega^2 = m_1/m_{-1}$, where $m_1 = \langle[F^+, [\mathcal{H}, F]]\rangle/2$ and $m_{-1} = -\chi(F)/2$ are two energy-weighted moments of the breathing operator $F = \cos\theta\sum_i x_{i\uparrow}^2 + \sin\theta\sum_j x_{j\downarrow}^2$, with a mixing angle $-\pi/2 < \theta < \pi/2$. The linear static response of the system $\chi$ can be calculated in terms of the susceptibility matrix $\partial^2 E_{\mathrm{hom}}/\partial n_\sigma \partial n_{\sigma'}$. Basically these two different breathing modes correspond to the in-phase and out-of-phase oscillations of the two imbalanced spin pop-

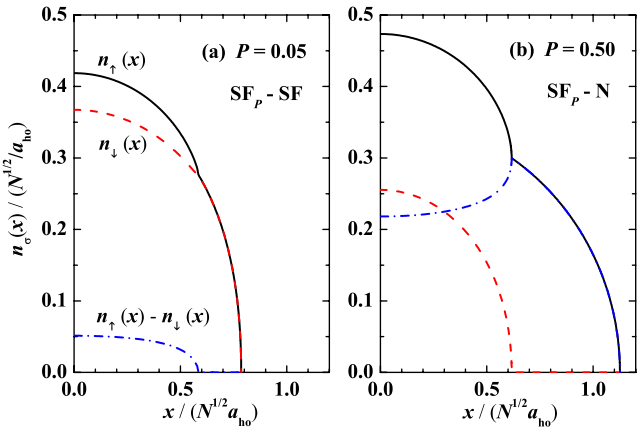


FIG. 3 (color online). Density profiles of each spin component and their difference at a typical interaction coupling of $Na_{1\mathrm{D}}^2/a_{\mathrm{ho}}^2 = 1$ for the $\mathrm{SF}_P$-SF phase (a) and the $\mathrm{SF}_P$-$N$ phase (b).

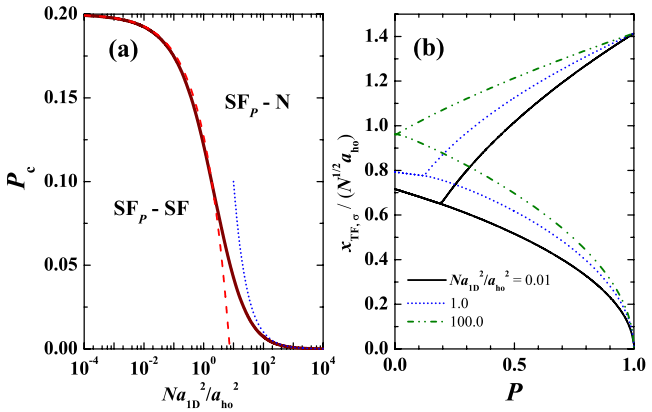


FIG. 4 (color online). (a) Phase diagram of an inhomogeneous 1D polarized Fermi gas. The dashed and dotted lines are asymptotic limits described in the text. (b) The Thomas-Fermi radius of each spin component as a function of the polarization at $Na_{1D}^2/a_{ho}^2 = 0.01$, 1, and 100.

ulations. They are decoupled for a mixing angle $\theta_{in} > 0$ or $\theta_{out} < 0$ that minimizes the mode frequency, analogous to the spin-charge separation in 1D. Then, the operators $F$ with these angles are anticipated to exhaust all the weights in the density and spin density channels, respectively. Thus, the sum-rule approach is well applicable, providing only an upper bound on the mode frequency. A stringent test of this single mode approximation merits further study, e.g., by using the random-phase approximation theory. Figure 5 shows the density breathing mode frequency as a function of polarization at several interaction strengths. With increasing polarization the frequency initially rises in the SF$_P$-SF state and then gradually decreases to the ideal gas result $2\omega_{ho}$ in the SF$_P$-$N$ phase. A peak structure is found that gives an independent means of identifying the FFLO phase, which dominates the phase diagram in the vicinity of this peak.

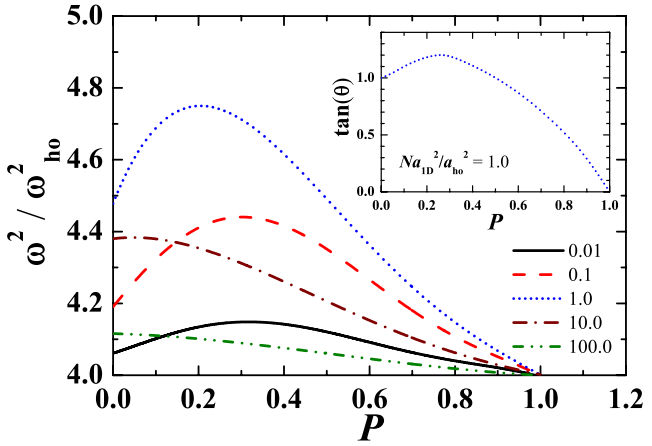


FIG. 5 (color online). Square of the lowest density breathing mode frequency, $\omega^2$, as a function of the spin polarization for interaction parameters as indicated. The inset shows the mixing angle that minimizes the sum-rule mode frequency at $Na_{1D}^2/a_{ho}^2 = 1.0$.

We emphasize that the FFLO superfluid reported here can be detected via a Josephson junction that is formed by confining two 1D polarized gases in a double-well trap. There is also a signature present in the density correlation function, which has an oscillatory behavior in this phase. Further details will be provided elsewhere.

In conclusion, we have investigated a 1D polarized Fermi gas in a harmonic trap, and have shown that the trap generally gives rise to phase separation: with at least one FFLO-type phase present. Two distinct exotic phase-separated structures can occur, and are detectable via absorption imaging and collective mode experiments.

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*Note added.*—After this work was completed, we became aware of a related work of G. Orso [18].

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