Statistics of Three-Dimensional Lagrangian Turbulence

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We consider a superstatistical model for a Lagrangian tracer particle in a high-Reynolds-number turbulent flow. The analytical model predictions are in excellent agreement with recent experimental data for flow between counter-rotating disks. In particular, the predicted Lagrangian scaling exponents ζ_j agree well with the measured exponents reported in H. Xu *et al.* [Phys. Rev. Lett. 96, 114503 (2006)]. The model also correctly predicts the shape of acceleration probability densities, correlation functions, statistical dependencies between components, and explains the fact that enstrophy lags behind dissipation.

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The full understanding of the statistical properties of fluid turbulence still remains a challenging problem in theoretical physics. In recent years there has been some experimental progress in measuring the statistical properties of single tracer particles advected by turbulent flows [1–9]. These experiments, as well as direct numerical simulations (DNS) [10–13], have significantly enhanced our knowledge of the stochastic properties of Lagrangian turbulence (LT). A variety of interesting experimental results have recently been published, in particular, for the probability densities of accelerations [1,2], velocity differences [3], correlation functions [4,5], conditional expectations [4,6,7], the Lagrangian scaling exponents ζ_j [5,7,8], and the corresponding $f(\alpha)$ spectra obtained by Legendre transformation [9]. Many of these new experimental data confirm early DNS results obtained in [10].

The exact theory of LT based on the Navier-Stokes equation still very much lags behind experimental progress. Hence it is important to develop simple theoretical models that provide an explanation for the most important statistical features of 3D LT. Previous models have been successful in explaining, e.g., the 1D measured acceleration statistics but fail to explain the recent experimental data for the 3D statistics [4] or the new data for the Lagrangian scaling exponents [8,9]. It is thus important to develop theoretical models that explain not just one but all of the above measured LT phenomena with sufficient precision at the same time, using a consistent set of parameters.

In this Letter we will introduce such a model, which for the first time simultaneously reproduces the measured 3D statistics, correlation functions, statistical dependencies between components, and scaling exponents. We will carefully compare its theoretical predictions with the available experimental data, obtaining excellent agreement. Our model is a natural and physically plausible extension of previous LT models based on superstatistical stochastic differential equations (SDEs) [14–20]. Here the term "superstatistics" [20] means that there is a superposition of several stochastic processes, a fast one as given by the original SDE and a slow one for the parameters of the SDE, which are regarded as slowly varying random variables describing the changing environment of the Lagrangian tracer particle. Our model approximates the high-Reynolds-number limit of a superstatistical extension of the Sawford model [15,21,22], and is more refined than previous models [14,15,17] by taking into account both a fluctuating energy dissipation and a fluctuating enstrophy surrounding the test particle.

We obtain predictions for Lagrangian scaling exponents ζ_j (and, by Legendre transform, for multifractal spectra) that are in very good agreement with the recent measurements of the Bodenschatz group [8,9]. The agreement seems to be better than for other models, e.g., the Lagrangian multifractal turbulence models compared with the data in [9]. At the same time our model correctly reproduces the measured probability densities of single velocity difference and acceleration components as well as those of the modulus, it describes correctly the fact that the three acceleration components are not statistically independent, it gives the correct conditional acceleration variance, and it explains the fact that the correlation function for single acceleration components decays rapidly whereas that of the modulus decays slowly. Finally, the model also explains why the fluctuating enstrophy lags behind the fluctuating energy dissipation, as numerically observed by Yeung and Pope [10] and experimentally by Zeff *et al.* [23]. To the best of our knowledge, our model is the first LT model that simultaneously achieves all this.

To introduce the model, let us denote the velocity of a Lagrangian tracer particle embedded in the turbulent flow by $\vec{v}(t)$. We are interested in velocity differences on given time scales τ, i.e., the quantity $\delta\vec{v}(t) = \vec{v}(t) - \vec{v}(t+\tau)$, which is directly measured in various experiments. To obtain a compact notation, in the following we write $\vec{u}(t) := \delta\vec{v}(t)$. Let us consider a linear superstatistical SDE for $\vec{u}(t)$ of the following form:

$$\dot{\vec{u}} = -\Gamma\vec{u} + \Sigma\vec{L}(t). \quad (1)$$

Here $\vec{L}(t)$ is a rapidly fluctuating stochastic process representing force differences in the liquid on a fast time scale, and Γ and Σ are 3×3 matrices. We approximate $\vec{L}(t)$ by

Gaussian white noise. $\Gamma(t)$ and $\Sigma(t)$ are matrix-valued stochastic processes which evolve on a much larger time scale than $\vec{L}(t)$.

Generally, the class of superstatistical models described by Eq. (1) is quite large. A first example was introduced in [14] and further developed in [15,17]. Let us here consider a more refined model that takes into account two very important facts: A fluctuating local energy dissipation rate of the environment surrounding the test particle and a fluctuating local enstrophy (rotational energy). We thus consider as a special case of Eq. (1) the local dynamics

$$\dot{\vec{u}} = -\gamma\vec{u} + B\vec{n} \times \vec{u} + \sigma\vec{L}(t). \tag{2}$$

We assume that γ and B are constants, but the noise strength σ and the unit vector $\vec{n}$ describing a temporary rotation axis of the particle evolve stochastically on a large time scale T_σ and $T_{\vec{n}}$, respectively. T_σ is of the same order of magnitude as the integral time scale T_L, whereas γ^{-1} is of the same order of magnitude as the Kolmogorov time scale τ_η. From the above one gets $T_\sigma\gamma \sim T_L/\tau_\eta \sim R_\lambda \gg 1$, i.e., the time scale separation between the slow and the fast processes, which is a necessary condition for the superstatistics approach to work [24], increases proportional to the Taylor scale Reynolds number R_λ. The time scale $T_{\vec{n}} \gg \tau_\eta$ describes the average life time of a region of given vorticity surrounding the test particle.

As it is customary in statistical physics, we define a parameter $\beta := 2\gamma/\sigma^2$, which in equilibrium statistical mechanics corresponds to the inverse temperature, whereas in superstatistical turbulence models [14–16] it is a formal parameter related to a suitable power ϵ^κ of the fluctuating energy dissipation rate ϵ. In the following we adopt the choice $\kappa = -1$, i.e., $\beta^{-1} \sim \nu^{1/2}\langle\epsilon\rangle^{-1/2}\epsilon$, where ν is the kinematic viscosity and $\langle\epsilon\rangle$ the average energy dissipation. To further specify our superstatistical model we still have to fix the probability density of the stochastic process $\beta(t)$, which, motivated by the cascade picture of turbulence and previous successful models [6,15,17,24–26], is assumed to be close to a log-normal distribution

$$f(\beta) = \frac{1}{\beta s\sqrt{2\pi}} \exp\left\{\frac{-(\log\frac{\beta}{m})^2}{2s^2}\right\}. \tag{3}$$

Here m and s are mean and variance parameters. The average β_0 of the above log-normal distribution is given by $\beta_0 = m\sqrt{w}$ and the variance by $\sigma^2 = m^2 w(w-1)$, where $w := e^{s^2}$.

On a time scale t satisfying $\gamma^{-1} \ll t \ll T_\sigma$ the probability density of a single component u_x of the tracer particle described by Eq. (2) is given by the Gaussian

$$p(u_x|\beta) = \sqrt{\frac{\beta}{2\pi}} e^{-(1/2)\beta u_x^2}. \tag{4}$$

Note that this result is independent of B and $\vec{n}$ [27]. In the long-term run the variance of this Gaussian will fluctuate since σ fluctuates. Hence we get a superposition of Gaussians with different variance parameter β^{-1}, i.e., the marginal stationary distribution of our superstatistical system is given by

$$p_{u_x}(u_x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \beta^{1/2} f(\beta) e^{-(1/2)\beta u_x^2} d\beta, \tag{5}$$

no matter what the value of B is. This formula, with $f(\beta)$ being the log-normal distribution, is in excellent agreement with experimentally measured histograms [1–3]. An example is shown in Fig. 1. One obtains good fittings of the data in [3] for all time scales τ, with $w \sim (\tau/\tau_\eta)^{-0.4}$. For very small τ the acceleration of the particle is given by $a_x = u_x/\tau$ and hence by transformation of random variables $p_{a_x}(a_x) = \tau p_{u_x}(u_x)$, thus

$$p_{a_x}(a_x) = \frac{\tau}{2\pi s} \int_0^\infty d\beta \beta^{-1/2} \exp\left\{\frac{-(\log\frac{\beta}{m})^2}{2s^2}\right\} e^{-(1/2)\beta\tau^2 a_x^2}. \tag{6}$$

This formula is in good agreement with the results presented in [1,2] ($s^2 = 3.0$, see [15] for details).

Let us now check what type of Lagrangian scaling exponents ζ_j for velocity increments u_x are predicted by our superstatistical model. From Eqs. (5) and (3) one obtains the moments

$$\langle u_x^j \rangle = (j-1)!! m^{-j/2} w^{(1/8)j^2}. \tag{7}$$

The notation $(j-1)!!$ stands for a product of all odd positive integers up to $j-1$. Assuming simple scaling laws of the form $m \sim \tau^a$, $w \sim \tau^b$, where a and b are so far arbitrary real numbers, Eq. (7) implies $\langle u_x^j \rangle \sim \tau^{-a(j/2)+b(1/8)j^2} \sim \tau^{\zeta_j}$. Hence

$$\zeta_j = -\frac{a}{2} j + \frac{b}{8} j^2. \tag{8}$$

It is often assumed that the Lagrangian exponent ζ_2 is equal to 1. From $\zeta_2 = 1$ we get $a = \frac{1}{2}b - 1$ thus

$$\zeta_j = \left(\frac{1}{2} + \lambda^2\right) j - \frac{1}{2}\lambda^2 j^2, \tag{9}$$

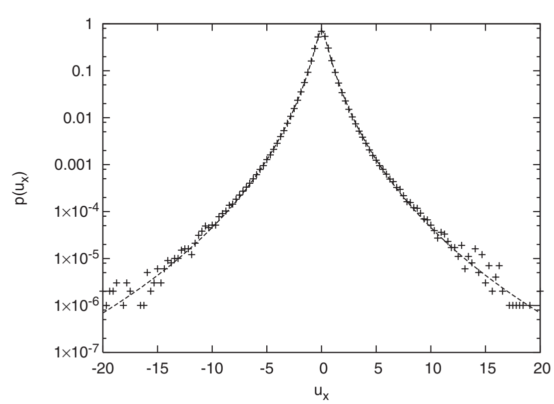

FIG. 1. Experimentally measured probability distribution $p(u_x)$ at smallest time scales (data from [3] for $\tau \approx 0.15$ ms) and a fit by Eqs. (3) and (5) with $s^2 = 1.8$.

where, following the notation of Mordant *et al.* [5] we defined a positive parameter λ^2 by $\lambda^2 = -\frac{1}{4}b$. Note that we get a formula analogous to Kolmogorov's 1962 theory (K62) [25]; however, the difference is that this formula is directly applicable to the Lagrangian dynamics, it needs not to be transformed from an Eulerian to a Lagrangian frame (as done in [9,28] for multifractal models). Our formula (9) is in very good agreement with the experimental data presented in [9], see Fig. 2. The agreement is better than for the other theoretical models compared with the data in [9]. Whereas all our predicted exponents are within the error bars of the experimental data, the predictions of the other models are outside the experimental error bars for $j \geq 5$ (dotted and dash-dotted lines in Fig. 2).

We may also proceed to the multifractal turbulence spectra $D(h)$ defined by $D(h) = \inf_j(hj + 1 - \zeta_j)$ by a Legendre transformation. The information contained in the $D(h)$ is the same as that in the ζ_j, and hence our predicted multifractal spectra obtained by Legendre transformation are again in good agreement with the experimental data presented in [9], better than the predictions of the other models.

Next, let us study the multivariate distribution $p(u_x, u_y, u_z)$ describing the joint probability distribution of the three components u_x, u_y, u_z of the Lagrangian particle. It is given by the superstatistical average

$$p(u_x, u_y, u_z) = \frac{1}{(2\pi)^{3/2}} \int_0^\infty \beta^{3/2} f(\beta) e^{-(1/2)\beta(u_x^2+u_y^2+u_z^2)} d\beta. \tag{10}$$

In particular, for small τ the distribution of the absolute value $|\vec{a}| = \tau^{-1}|\vec{u}|$ of acceleration is given by

$$p(|\vec{a}|) = 4\pi|\vec{a}|^2 p(a_x, a_y, a_z) \tag{11}$$

$$= \sqrt{\frac{2}{\pi}}|\vec{a}|^2\tau^3 \int_0^\infty \beta^{3/2} f(\beta) e^{-(1/2)\beta\tau^2|\vec{a}|^2} d\beta. \tag{12}$$

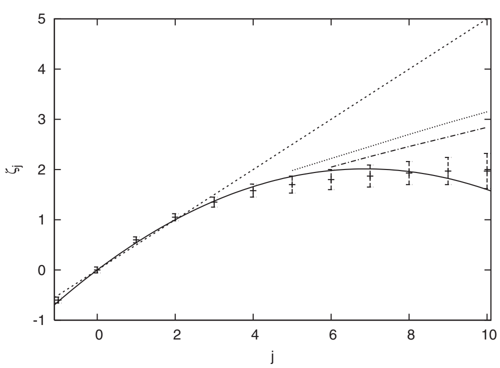

FIG. 2. Lagrangian scaling exponents ζ_j as measured by Xu *et al.* [9] and as predicted by the superstatistical model for $\lambda^2 = 0.085$ (solid line). Other model predictions are also shown (from top to bottom): no intermittency at all (dashed line), Lagrangian version of She-Lévêque model [9,29] (dotted line), and multifractal model of Chevillard *et al.* [9,28] (dash-dotted line).

The agreement of this formula with experimentally measured distributions of the acceleration modulus has been checked in [6], taking again for $f(\beta)$ the log-normal distribution. Excellent agreement was found. Note that in Eq. (10) the 3-point density is not the product of 1-point densities as given by Eq. (5), and hence the model naturally introduces statistical dependence between the acceleration components.

We may investigate this effect in a quantitative way, by studying the ratios $R := p(a_x, a_y)/[p(a_x)p(a_y)]$. From our superstatistical 3D model we obtain the general prediction

$$R = \frac{\int_0^\infty \beta f(\beta) e^{-(1/2)\beta\tau^2(a_x^2+a_y^2)} d\beta}{\int_0^\infty \beta^{1/2} f(\beta) e^{-(1/2)\beta\tau^2 a_x^2} d\beta \int_0^\infty \beta^{1/2} f(\beta) e^{-(1/2)\beta\tau^2 a_y^2} d\beta}, \tag{13}$$

which is plotted in Fig. 3 for the example of the log-normal distribution $f(\beta)$. One obtains a diagram that closely resembles the experimentally measured data (Fig. 5 in [6]).

All kinds of quantities describing the statistical dependence can be analytically evaluated for our model. For example, one obtains the conditional moments $\langle a_x^j | a_y \rangle$ as

$$\langle a_x^j | a_y \rangle = (j-1)!!\tau^{-j} \frac{\int_0^\infty d\beta \beta^{(1-j)/2} f(\beta) e^{-(1/2)\beta\tau^2 a_y^2}}{\int_0^\infty d\beta \beta^{1/2} f(\beta) e^{-(1/2)\beta\tau^2 a_y^2}} \tag{14}$$

for j even (they vanish for j odd). Moreover, for log-normal superstatistics one obtains

$$\frac{\langle a_x^i a_y^j \rangle}{\langle a_x^i \rangle \langle a_y^j \rangle} = w^{(1/4)ij} = e^{(1/4)ijs^2}, \tag{15}$$

(i, j even) which yields a relation between the flatness parameter w and the statistical dependencies of the acceleration components that can be checked in future experiments.

Our model also allows for the calculation of temporal correlation functions. In particular we may be interested in temporal correlation functions of single components u_x of velocity differences, i.e., $C(t) = \langle u_x(t' + t) u_x(t') \rangle$. By averaging over the possible random vectors $\vec{n}$ one arrives

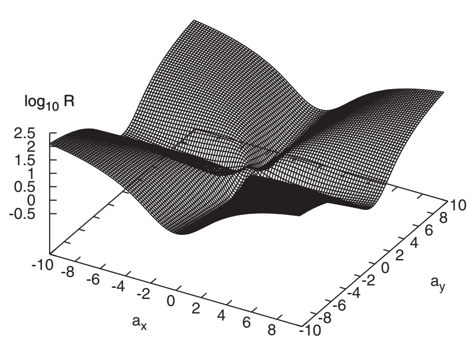

FIG. 3. The ratio $R = p(a_x, a_y)/[p(a_x)p(a_y)]$ as predicted by the superstatistical model for $s^2 = 3$. $R \neq 1$ indicates statistical dependence of acceleration components.

at the formula

$$C(t) = \frac{1}{3}\langle u_x^2\rangle e^{-\gamma t}(2\cos Bt + 1), \tag{16}$$

i.e., there is rapid (exponential) decay with a zero crossing at $t^* = \frac{2}{3}\pi B^{-1}$. Exponential decay and zero crossings are indeed observed for the experimental data [4,5,7] as well as in Lagrangian DNS [12]. The experimentally observed zero of the correlation function [4,5] can be used to estimate the size of the parameter B. DNS [10,12] indicates that $t^* \approx 2.2\tau_\eta$ independent of R_λ, thus our model parameter B is given by $B \approx 0.95\tau_\eta^{-1}$.

Higher-order correlation functions are of interest, too. For example, the correlation function of the square of velocity differences $\hat{C}(t) = \langle \vec{u}^2(t'+t)\vec{u}^2(t)\rangle$ can be approximated as $\hat{C}(t) \approx \langle \beta^{-1}(t'+t)\beta^{-1}(t')\rangle$. Clearly, by construction of the superstatistical model, this correlation function decays very slowly, since the process $\beta(t)$ evolves on a much larger time scale T_σ than the process $u(t)$. This makes it clear why the correlation function of the modulus $|\vec{u}| = \sqrt{|\vec{u}|^2}$ decays very slowly, as it is indeed observed in experiments [6,7] and in DNS [10].

Finally, let us comment on the typical evolution of dissipation and enstrophy fluctuations in our model. A large value of local energy dissipation ϵ corresponds to a large value of σ^2, since in our superstatistical dynamical model $\epsilon \sim \beta^{-1} \sim \sigma^2/(2\gamma)$. This means the forcing $\sigma\vec{L}(t)$ is strong for a while. After some short relaxation time of the order $\gamma^{-1} \sim \tau_\eta$, where τ_η is the Kolmogorov time, this will create a large local acceleration variance $\langle a^2\rangle \sim \langle u^2\rangle\tau^{-2} \sim \beta^{-1}\tau^{-2}$. The term $B\vec{n}\times\vec{u}$ will then create a lot of rotational energy (= enstrophy Ω), as soon as $|\vec{u}|$ has become large. Thus energy dissipation and enstrophy are strongly correlated in time, and enstrophy lags behind dissipation evolution by something of the order τ_η (the relaxation time of the system). This is actually experimentally observed in Fig. 1 of [23]. The peaks of ϵ and Ω are shifted by about a half of a second, which corresponds to the Kolmogorov time of the system studied by Zeff *et al.* [23]. Our simple superstatistical model describes these effects in a correct way. The above time-lag effect also shows up as an asymmetry of the dissipation-enstrophy cross-correlation function in DNS [10,12].

To conclude, we have demonstrated that the most important statistical phenomena that have been experimentally reported in LT experiments so far are well reproduced by a superstatistical model that can be regarded as a generalized Brownian motion model relevant for 3D Lagrangian tracer dynamics. The model arises out of a physically plausible local momentum balance equation for the Lagrangian particle, and, compared to other recent Lagrangian models [30], has the advantage of being analytically treatable. The model naturally incorporates a superposition of several stochastic processes, a fast one for velocity differences of the tracer particle and slow ones for dissipation and enstrophy in the environment of the tracer particle. The predicted Lagrangian scaling exponents ζ_j, the 1-point and 3-point probability densities, correlation functions, as well as the statistical dependencies between acceleration components are in excellent agreement with the experimental data.

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