Fidelity and Quantum Chaos in the Mesoscopic Device for the Josephson Flux Qubit

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We show that the three-junction SQUID device designed for the Josephson flux qubit can be used to study the dynamics of quantum chaos when operated at high energies. We determine the parameter region where the system is classically chaotic. We calculate numerically the fidelity or Loschmidt echo (LE) in the quantum dynamics under perturbations in the magnetic field and in the critical currents, and study different regimes of the LE. We discuss how the LE could be observed experimentally considering both the preparation of the initial state and the measurement procedure.

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Ultrasmall Josephson devices have been used as tools for studying quantum phenomena at the macroscopic level since the 1980s [1]. Macroscopic quantum tunneling [2] and macroscopic quantum coherence of flux [3] and persistent currents [4] have been observed experimentally. More recently, mesoscopic Josephson devices have been used for the design of qubits for quantum computation [5–8]. The progress made in this case allows us to have nowadays Josephson circuits with small dissipation and large decoherence times, giving place to a coherent manipulation of the system [7,8]. This could also make possible the use of Josephson devices for the study of the quantum dynamics of chaotic systems, a subject where there has been a great interest in the last years [9–14].

The stability of the quantum dynamics of a system against perturbations [9] can be quantified by the fidelity or Loschmidt echo (LE) [10]. The LE is the overlap between two states that evolve from the same initial wave function $|\Psi_0\rangle$ under two slightly different Hamiltonians,

$$F(t) = |f(t)|^2 = |\langle\Psi_0|e^{iH_\varepsilon t/\hbar}e^{-iH_0 t/\hbar}|\Psi_0\rangle|^2, \qquad (1)$$

with H_0, H_ε the unperturbed and perturbed Hamiltonians, respectively. In classically chaotic systems, above a perturbative regime where the LE has a Gaussian decay for short times, the LE shows an exponential decay for large t, $F(t) \propto e^{-\gamma t}$ [10–14]. For weak perturbation strength ε the decay constant γ depends as $\gamma(\varepsilon) \propto \varepsilon^2$, and is obtained from the Fermi golden rule (the FGR regime) [11], while for strong perturbations γ becomes independent of ε and saturates at the classical Lyapunov exponent λ (the Lyapunov regime) [10]. There have been some recent experimental measurements of the LE [15–17]. Here we will show that the device for the Josephson flux qubit (DJFQ) studied in [6–8] is also a promising system for the experimental observation of the LE.

The DJFQ consists of three Josephson junctions in a superconducting ring [6] that encloses a magnetic flux $\Phi = f\Phi_0$, with $\Phi_0 = h/2e$. Two of the junctions have the same coupling energy E_J and capacitance C, while the third junction has couplings αE_J and αC, respectively ($0.5 < \alpha < 1$). Typically the circuit inductance can be neglected and the phase difference of the third junction is $\varphi_3 = 2\pi f + \varphi_1 - \varphi_2$, leading to the Hamiltonian [6]

$$\mathcal{H} = \tfrac{1}{2}\vec{P}^T\mathbf{M}^{-1}\vec{P} + E_J V(\vec{\varphi}), \qquad (2)$$

where $\vec{\varphi} = (\varphi_1, \varphi_2)$, $\vec{P} = \mathbf{M}\cdot d\vec{\varphi}/dt$, and

$$\mathbf{M} = \left(\frac{\Phi_0}{2\pi}\right)^2 C\begin{pmatrix} 1+\alpha+\gamma & -\alpha \\ -\alpha & 1+\alpha+\gamma \end{pmatrix} = \frac{\hbar^2}{\eta^2 E_J}\mathbf{m}.$$

Here $\eta = \sqrt{8E_C/E_J}$ with $E_C = e^2/2C$, we include in $\mathbf{M}$ the on-site gate capacitance $C_g = \gamma C$, and

$$V(\vec{\varphi}) = 2 + \alpha - \cos\varphi_1 - \cos\varphi_2 - \alpha\cos(2\pi f + \varphi_1 - \varphi_2). \qquad (3)$$

In the quantum regime, $\hat{\vec{P}} = -i\hbar\vec{\nabla}_\varphi = -i\hbar(\frac{\partial}{\partial\varphi_1}, \frac{\partial}{\partial\varphi_2})$, the time-dependent Schrödinger equation is

$$i\eta\frac{\partial\Psi(\vec{\varphi})}{\partial t} = \left[-\frac{\eta^2}{2}\vec{\nabla}_\varphi^T\mathbf{m}^{-1}\vec{\nabla}_\varphi + V(\vec{\varphi})\right]\Psi(\vec{\varphi}), \qquad (4)$$

where we normalized time by $t_c = \hbar/\eta E_J$, energy by E_J, and momentum by $\hbar/\eta$. We see in Eq. (4) that the parameter η plays the role of an effective $\hbar$. For quantum computation implementations [6–8] the DJFQ is operated at magnetic fields near the half-flux quantum ($f = 1/2 + \delta f$). In this case the two lowest energy eigenstates are symmetric and antisymmetric superpositions of two states corresponding to macroscopic persistent currents of opposite sign. These two eigenstates are energetically separated from the others (for small δf) and therefore the DJFQ has been used as a qubit [6–8]. As we will discuss here, the higher energy states of the DJFQ show quantum manifestations of classical chaos.

It has been found in [18,19] that superconducting loops with three Josephson junctions and on-site capacitances ($\gamma = \infty$) can be chaotic. Most Josephson circuits, like the DJFQ, have small on-site gate capacitances ($\gamma \sim 10^{-2}$). Here we analyze the dynamics of the DJFQ considering the realistic case with $\gamma = 0.02$ [6]. We first obtain the classical dynamical evolution integrating the Hamilton equations that correspond to Eq. (2) with a second order Verlet algorithm. For different values of the parameter α

and magnetic field f we compute the maximum Lyapunov exponent λ for each orbit at different energies E. We estimate the chaotic volume $v_{ch}(E)$, as the probability of having a chaotic orbit (i.e., $\lambda > 0$) for a given E, using 10^3 initial conditions randomly chosen with uniform probability within the available phase space. Also the average Lyapunov exponent $\bar{\lambda}(E)$ of the chaotic orbits is obtained. In Fig. 1(a) we show $v_{ch}(E)$ and $\bar{\lambda}(E)$ for $f = 0.5$ and $\alpha = 0.8$. Above the minimum energy of the potential, E_{min}, we find (i) regular orbits for $E_{min} < E < E_{ch}$ ($v_{ch} = 0$), (ii) soft chaos (i.e., coexistence of regular and chaotic orbits, $0 < v_{ch} < 1$) for $E_{ch} < E < E_{hc}$, a Poincaré section for this case is shown in the inset of Fig. 1(a), and (iii) hard chaos (all orbits are chaotic, $v_{ch} = 1$) for $E > E_{ch}$. The average Lyapunov exponent is $\bar{\lambda} > 0$ above E_{ch}. In Fig. 1(b) we show the energy boundaries of the different regimes (E_{min}, E_{ch}, E_{hc}) as a function of $\delta f = f - 1/2$ for $\alpha = 0.8$. We can see that for $f = 1/2$ the onsets of soft and hard chaos, E_{ch}, E_{hc}, are closer to E_{min} in comparison with other values of f. In the inset of Fig. 1(b) we also show the boundaries of the dynamic regimes as a function of α for $f = 1/2$. The Hamiltonian dynamics of Eq. (2) are a good approximation of the problem for energies $E < 2\Delta$, with Δ the superconducting gap. The $Al/AlO_x/Al$ junctions of [6] have $2\Delta \approx 3.7E_J$. Since we find $E_{hc} = 1.75E_J$, there is a wide energy range where the hard chaos regime is experimentally accessible. Furthermore, we find that for realistic experimental parameters ($\eta = 0.1 - 0.5$, $\alpha = 0.7 - 0.8$), the third quantum energy level can be above the classical

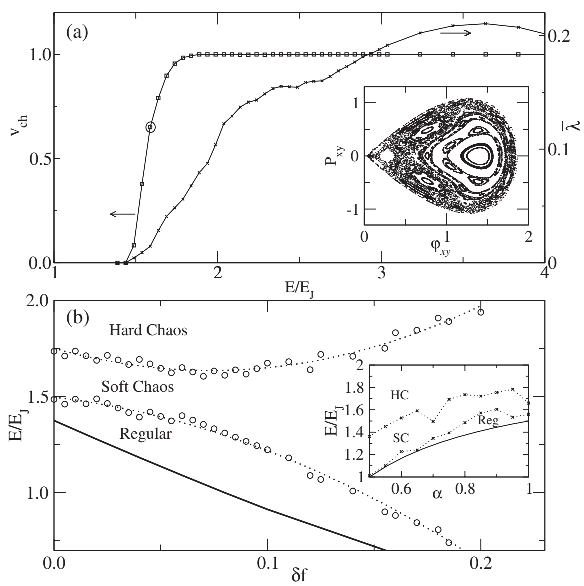

FIG. 1. (a) Chaotic volume v_{ch} and average Lyapunov exponent $\bar{\lambda}$ versus energy E for $\alpha = 0.8$ and $f = 1/2$. Inset: surface of section $\varphi_{xy} = (\varphi_1 - \varphi_2)/\sqrt{2}$, $P_{xy} = (P_1 - P_2)/\sqrt{2}$ at $E = 1.6E_J$. (b) Energy boundaries for the regimes of regular orbits, soft chaos and hard chaos as a function of $\delta f = f - 1/2$ for $\alpha = 0.8$. The continuous line corresponds to the potential energy minimum, E_{min}. Inset: Energy boundaries for the different regimes as a function of α for $f = 1/2$.

onset of chaos, E_{ch}, for $f = 1/2$. In Ref. [19], for the case with $\gamma = \infty$, an analysis of the statistics of the energy spectra of the quantum Hamiltonian shows that it belongs to the Gaussian orthogonal ensemble in the hard chaos regime. We find a similar result in our case, with $\gamma = 0.02$ and $f = 1/2$, for $E > E_{hc}$ [20].

We now calculate the quantum dynamics of the DJFQ integrating numerically Eq. (4) with a fourth order split-operator algorithm as in [21], with a discretization grid of $\Delta\varphi = 2\pi/128$ and $\Delta t = 0.015\eta t_C$. We use 2π-periodic boundary conditions on $\vec{\varphi} = (\varphi_1, \varphi_2)$. To compute the LE of Eq. (1), the simulations are started from minimum-uncertainty 2π-periodical wave packets [22] given by

$$|\Psi_0\rangle = \Psi_{\vec{K}_0,\vec{\varphi}_0}(\vec{\varphi}) = Ce^{i\vec{K}_0(\vec{\varphi}-\vec{\varphi}_0)}e^{-B(\vec{\varphi}-\vec{\varphi}_0)/2\sigma^2}, \quad (5)$$

with $B(\vec{\varphi}) = 2 - \cos\varphi_1 - \cos\varphi_2$, $[B(\vec{\varphi}) \approx |\vec{\varphi}|^2/2$ for small $|\vec{\varphi}|]$, $\vec{K}_0 = (k_1, k_2)$ with k_1, k_2 integers, and σ is the width of the wave packet. The LE is usually computed as the average $\bar{F}(t)$ over different $|\Psi_0\rangle$ (see [12,14]). Here we average over 15 initial conditions with different $\vec{K}_0$, $\vec{\varphi}_0$ corresponding to the same classical energy. We choose $\sigma^2 = 0.31\eta$, which corresponds to a spectral width of $\Delta E \approx 0.3E_J$. We consider $E = 3E_J > E_{hc}$, for which the classical phase space is filled by a connected region of chaos with Lyapunov exponent $\lambda = 0.182 \pm 0.008$.

We evaluate the LE of the DJFQ against perturbations in the parameters α and f. In Fig. 2(a) we show the time dependence of $\bar{F}(t)$ for different perturbations in the parameter $\alpha = \alpha_0 + \delta\alpha$ for $\alpha_0 = 0.8$, while in Fig. 2(b) we show $\bar{F}(t)$ for different perturbations in $f = 1/2 + \delta f$. We can see that in both cases $\bar{F}(t)$ decays with time, and that the decay rate tends to increase when increasing the perturbation. Above a crossover value of the perturbation ($\delta\alpha_c$ or δf_c) we find that the curves of $\bar{F}(t)$ tend to overlap. The overall decay of the LE has been described with the form $\bar{F}(t) \sim Ae^{-\gamma t} + F_\infty$ [12]. For large perturbations the decay rate γ saturates at a value close to the Lyapunov exponent λ of the classical dynamics. This can be seen in Fig. 2 where the dashed lines show the slope of a decay rate with the Lyapunov exponent for comparison. The constant value F_∞ is proportional to the inverse of the fraction of the volume of the Hilbert space spanned by the initial wave function $|\Psi_0\rangle$ [12]. In our case this corresponds to $F_\infty \propto \sigma^2/(2\pi)^2 = 0.31\eta/(2\pi)^2 \approx 0.0013$ for $\eta = 0.17$, which is close to the results of Figs. 2(a) and 2(b). An analysis of the variance of the fidelity, $\sigma_F^2(t) = \overline{F(t)^2} - \overline{F(t)}^2$ is also important [14]. This is shown in the insets of Fig. 2. We find that for increasing perturbations σ_F^2 saturates to a decay given by $\sigma_F^2(t) \sim e^{-2\lambda t}$, as discussed in [14].

We obtain the decay rate γ fitting the exponential form $Ae^{-\gamma t} + F_\infty$ for $\bar{F}(t)$ for times above the initial Gaussian decay (we have chosen $t > 15t_J$ in this case). Figure 3(a) shows the obtained γ as a function of the perturbation δf for different values of η. For small perturbations we obtain a quadratic law dependence of the decay rate with the perturbation strength $\gamma \propto (\delta f)^2$, which corresponds to

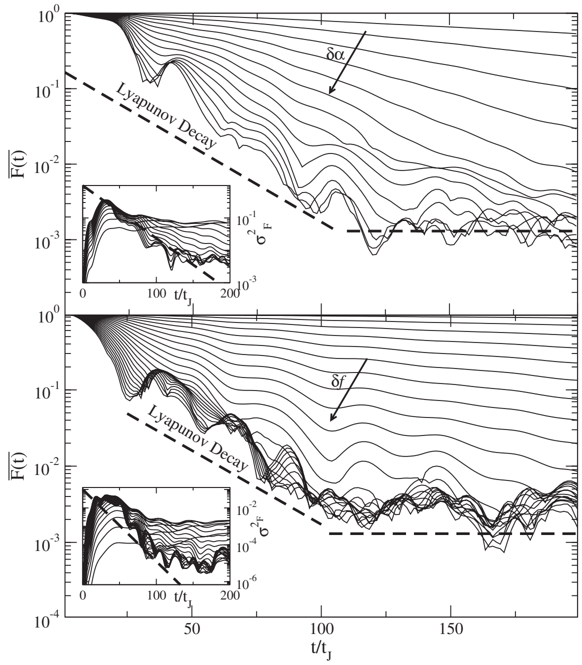

FIG. 2. The average Loschmidt echo $\bar{F}(t)$ as a function of time for different values of the perturbation strength of (a) $\delta\alpha$ and (b) δf. The energy is $E \simeq 3E_J$, deep in the hard chaos region, and $\hbar_{\rm eff} \equiv \eta = \sqrt{8E_C/E_J} = 0.17$. Time is measured in units of $t_J = \hbar/E_J = \eta t_C$. Insets: time evolution of $\sigma_F^2 = \overline{F^2(t)} - \overline{F(t)}^2$. The dashed lines represent an exponential decay given by the Lyapunov exponent ($\lambda = 0.182 \pm 0.008$) with the $t = 0$ offset shifted for clarity.

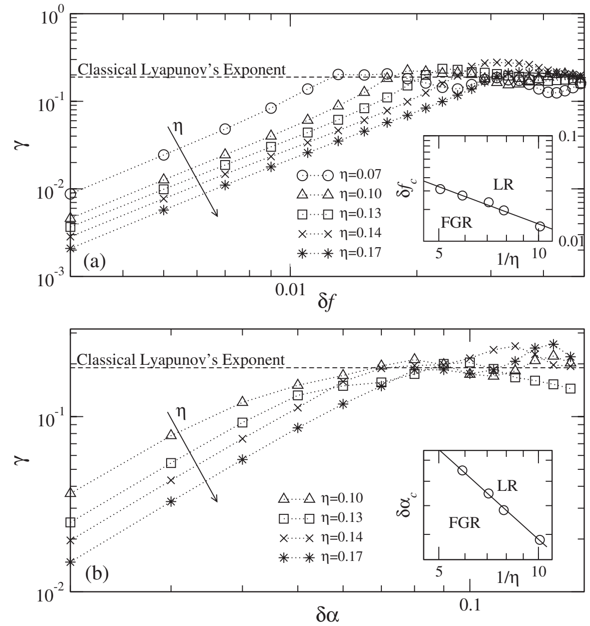

FIG. 3. (a) Decay rate of the LE, γ, as a function of the perturbation strength δf for different values of η, with $E \simeq 3E_J$. The classical Lyapunov exponent is indicated in the dashed line. Inset: crossover perturbation δf_c as a function of η^{-1}. (FGR: Fermi golden rule regime, LR: Lyapunov regime). (b) Same as (a) for perturbations $\delta\alpha$.

the Fermi Golden Rule (FGR) regime [11]. For large perturbations the obtained values of γ have a large error, which is of the size of the oscillations seen in the data in Fig. 3(a) for large δf. However, when comparing different cases of η, we see that for large perturbations the values of γ fall close to the Lyapunov exponent (shown as a dashed line). We obtain an estimate of the crossover value δf_c where γ saturates to λ. In the inset of Fig. 3(a) we see the dependence of δf_c with η. We find that δf_c decreases for decreasing η since quantum fluctuations become less important, and therefore the classical Lyapunov decay is reached more easily. We find a similar behavior for the dependence of γ with $\delta\alpha$ and η [shown in Fig. 3(b)].

Therefore, we have shown that the DJFQ at high energies shows a decay of the LE, similar to other chaotic systems [10–14]. This behavior could be experimentally observable if the time scale for the Lyapunov decay, $\tau_{\rm Lyap}$ is much smaller than the decoherence time $\tau_{\rm decoh}$ due to external sources. We estimate $\tau_{\rm Lyap} = \hbar/(\eta E_J \lambda) \sim$ 0.04–0.2 ns, using $\lambda \approx 0.18$ from the hard chaos regime and experimental parameters [7,8]. In [7] a value of $\tau_{\rm decoh} \approx 20$ ns was obtained for the lowest energy states. At high energies ($E \sim 3E_J$), we find that the spectrum is 1 order of magnitude more dense than at low energies, therefore $\tau_{\rm decoh}$ should be at least 1 order of magnitude smaller, $\tau_{\rm decoh} \sim 1$ ns. These roughly estimated values of $\tau_{\rm Lyap} \sim 0.1$ ns and $\tau_{\rm decoh} \sim 1$ ns leave some room for observing the FGR and Lyapunov regimes of the LE. However, in order to realize an experiment, the following two issues have to be solved: (i) preparation of the initial state and (ii) measurement of the fidelity $F(t)$.

(i) Preparation of the initial state.—To observe the Lyapunov regime the system has to be started from a wave packet narrowly localized in both coordinate (phase) and momentum (charge) [10,11], as in Eq. (5). Here we suggest the procedure shown schematically in Fig. 4(a). To localize the momentum (charge): each of the junctions is connected in parallel to a voltage source V_s, which builds up a charge in each junction. (For $E = 3E_J$ a voltage of $V_s \approx 0.1$ mV is needed.) To localize the coordinate (phase): each of the nodes of the DJFQ is connected to a large superconducting reservoir through a dc SQUID as in [23]. Then the system is prepared staying with $V_s \neq 0$ and $\Phi_{\rm SQUID} = 0$ for a long time ($t < 0$), and at $t = 0$ the voltage sources are set to $V_s = 0$ and the flux in the SQUIDs to $\Phi_{\rm SQUID} = \Phi_0/2$.

(ii) Measurement of the fidelity.—One has to be able to measure the overlap of Eq. (1). One protocol proposed originally in [24] and applied in [25,26] (see also [15]) consists in coupling the chaotic system under consideration with a qubit that acts both as a perturbing and a measuring device. The approximate Hamiltonian for this case is $H = H_0 \otimes |0\rangle\langle 0| + H_\epsilon \otimes |1\rangle\langle 1|$. H_0, H_ϵ are the Hamiltonians of the unperturbed and perturbed system, respectively, and $|0\rangle$, $|1\rangle$ are the two basis states of the qubit such that when

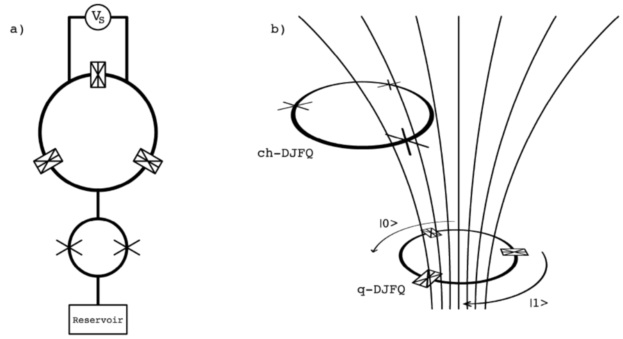

FIG. 4. Schematic setup for the observation of the Loschmidt echo in a DJFQ. (a) Circuit for preparation of the initial wave packet. V_s is a voltage source and the SQUID connects a node of the DJFQ with a superconducting reservoir. Only one voltage source and one SQUID are drawn, for simplicity. (b) Layout for the measurement of the Loschmidt echo. ch-DJFQ: circuit with quantum evolution in the chaotic regime; q-DJFQ: circuit used as a qubit for measurement of the LE.

the qubit is in the $|1\rangle$ state it induces a perturbation ϵ in the chaotic system. A Ramsey type of experiment is performed: after a time t, $\pi/2$ pulses are applied to the qubit for two evolutions from initial states of the system $|\Psi_0\rangle \otimes (|0\rangle + a|1\rangle)/\sqrt{2}$ with $a = 1$ and $a = i$, respectively, from which the fidelity amplitude $f(t)$ can be obtained; see [24–26] for details. A possible implementation of this idea is shown in Fig. 4(b). A second DJFQ could be used operating in the "qubit' regime (called q-DJFQ) for measurement of the LE in a DJFQ evolving in the quantum "chaotic" regime (called ch-DJFQ). The q-DJFQ is coupled inductively to the ch-DJFQ, such that when the q-DJFQ is in the $|0\rangle$ ($|1\rangle$) state a clockwise (counterclockwise) current flowing in it induces a positive (negative) perturbation in the magnetic flux threaded by the ch-DJFQ. The q-DJFQ should have a smaller area than the ch-DJFQ (such that the flux induced in the q-DJFQ by the currents in the ch-DJFQ is small). For a better observation of the LE the ch-DJFQ should be built in a more semiclassical regime ($\eta \sim 0.1$, for example), while the q-DJFQ is built in a more quantum regime ($\eta \sim 0.5$, for example). Furthermore, making measurements with the measuring q-DJFQ placed at different distances from the ch-DJFQ could allow to obtain the LE for different perturbation intensities. For example, if when placing the q-DJFQ closer to the ch-DJFQ the decay of the LE becomes independent of the distance, that would indicate that the Lyapunov regime was reached. The state preparation and measurement procedure described here requires coupling to several external objects, and thus the resulting decoherence rates will inevitably increase. However, it has been shown in general grounds that $\tau_{\mathrm{Lyap}} \leq \tau_{\mathrm{decoh}}$ [13]. This implies that at least the Lyapunov regime of the LE will be observable, and for this reason it is already interesting to perform the experiment. Moreover, one advantage of Josephson nanocircuits is that they can be fabricated with well-controlled parameters allowing to study the LE for different cases of the effective $\hbar$ ($\hbar_{\mathrm{eff}} \equiv \eta$).

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