# Elastic Composite Materials Having a Negative Stiffness Phase Can Be Stable

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We prove that composite materials containing an isotropic phase having negative bulk and Young's moduli (hence being unstable by itself) can be stable overall, under merely applied traction boundary conditions, if the stable encapsulating phase is sufficiently stiff. We derive specific quantitative requirements on the elastic moduli of the constituent materials that ensure composite stability for two fundamental composite geometries. These results legitimize the concept of negative-stiffness-phase composites, thus dramatically expanding the parameter landscape in which novel and optimal overall material properties may be sought.

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Composite materials are constructed by combining two or more materials in such a manner that the overall properties of the result are superior in some desired way to those of the constituents individually. The attainable overall stiffnesses of elastic composites are restricted by rigorous bounds [1,2] which had been taken as inviolate for nearly the past half-century; they are based on the assumption that all constituent materials have positive stiffnesses (positive-definite elastic modulus tensors)—e.g., for isotropic phases, their tensile, shear, and bulk moduli are all positive. However, it was recently shown experimentally [3] and theoretically [4] that relaxing this assumption—permitting one phase of the composite to have an appropriately tuned negative stiffness—can produce composite stiffnesses that dramatically exceed these bounds, and, in fact, are predicted theoretically to exceed the stiffnesses of any known material [4]. Further, experiments [3] and theory [5,6] have shown that an appropriately tuned negative-stiffness phase can produce composite damping, thermal expansion, piezoelectricity, and pyroelectricity far exceeding all standard bounds. These works are indicative, but far from exhaustive, illustrations of the fascinating possibilities for novel materials if the use of unstable phases is countenanced.

The key outstanding question is whether a composite material containing an unstable phase can be stable overall, especially in the practically important case of prescribed load boundary conditions, since it is well known that a negative-stiffness material is unstable by itself under such boundary conditions.

Here we answer this question by explicitly proving for the first time that elastic composite materials can be stable even if they contain an encapsulated phase that has merely strongly elliptic moduli. This legitimizes relaxation of the positive-definiteness requirement on all phases when seeking composite materials having novel and optimal properties, thus dramatically expanding the parameter landscape. [Strong ellipticity means the fourth-rank elastic modulus tensor $\mathbf{C}$, which relates the stress and strain tensors as $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$, must satisfy $(\mathbf{ab}) : \mathbf{C} : (\mathbf{ab}) > 0$ for all dyads $\mathbf{ab} \neq \mathbf{0}$; this permits negative bulk and Young's moduli in the isotropic material case, as shown explicitly later in Eq. (13). Positive definiteness is the much stronger requirement $\boldsymbol{\alpha} : \mathbf{C} : \boldsymbol{\alpha} > 0$ for all symmetric second-rank tensors $\boldsymbol{\alpha} \neq \mathbf{0}$.]

The method of proof employed is direct and novel: while typical stability analyses employ multiple-infinite-series representations in all space variables of the perturbing field (here, the displacement vector field), we show that use of an energy approach together with key calculus analysis of the general sufficient condition for stability facilitates a stability analysis in which only one displacement component must be expanded in a single-variable Fourier series. The analysis is first performed for a two-dimensional composite, where the mathematics is relatively simple and transparent; we then show the same conceptual approach also works for a three-dimensional composite.

A sufficient condition for stability is that in any geometrically possible displacement perturbation from an equilibrium position, the internal energy stored or dissipated must exceed the work done on the system by the external loads [7]. This condition ensures stability in the dynamic Liapunov sense if the physically sensible exclusion of unbounded strain gradients is made, the perturbing displacement field and its gradient are sufficiently small, and the body is supported against rigid-body motion [8].

We consider stability of the equilibrium unstrained (ground) state of an elastic composite material when subject to arbitrary infinitesimal geometrically possible displacement field perturbations, since our objective is to examine material stability. For an isotropic elastic body subject to dead loads (static loads that do not vary during an arbitrary infinitesimal displacement), the above-stated sufficient condition for stability is [7,8]:

$$\int_V [2\mu\boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} + \lambda(\mathrm{tr}\boldsymbol{\varepsilon})^2] dV > 0, \tag{1}$$

where $V$ is the volume of the body, $\mu$, $\lambda$ are the Lamé elastic moduli, $\mu$ being the shear modulus, and $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor ( : denoting scalar product and tr the trace) associated with the arbitrary (but not identically zero) kinematically admissible *perturbing* displacement

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field $\mathbf{u}$ as $\boldsymbol{\varepsilon} = \mathrm{sym}(\mathbf{u}\boldsymbol{\nabla})$, the symmetric part of the displacement gradient field. For simplicity and explicitness, we will consider composite materials comprised of homogeneous, isotropic phases.

We analyze two fundamental composite geometries, illustrated in Fig. 1: the two-dimensional (plane strain) problem of an infinitely long circular cylinder of non-positive-definite material coated with a thin layer of positive-definite material; and the three-dimensional problem of a non-positive-definite sphere with a thin positive-definite coating. In each case, the inclusion has radius $a$, volume $V_1$, surface $S_1$, and elastic moduli $\lambda_1$ and $\mu_1$; and the layer has thickness $t$, volume $V_2$, and elastic moduli $\lambda_2$ and $\mu_2$. We emphasize that *arbitrary* infinitesimal kinematically admissible perturbing displacement fields are treated; for example, neither axial symmetry (in the 2D problem) nor spherical symmetry (in the 3D case) are assumed.

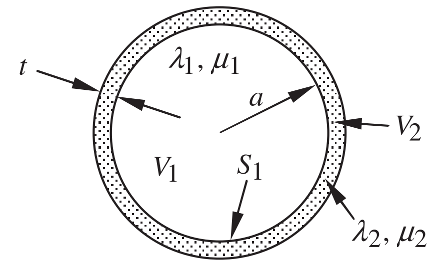


FIG. 1. Geometry and pertinent quantities for the two- and three-dimensional problems analyzed.

The stability condition, Eq. (1), can be rewritten, applicable to both the 2D (coated cylinder, $k = 2$) and 3D (coated sphere, $k = 3$) problems explained above, as

$$\int_{V_1}\left\{2\mu_1[(1-\beta)\boldsymbol{\varepsilon}':\boldsymbol{\varepsilon}' + \beta\boldsymbol{\omega}:\boldsymbol{\omega}] + \left(\lambda_1 + \frac{2\mu_1}{k}[1+(k-1)\beta]\right)(\mathrm{tr}\boldsymbol{\varepsilon})^2\right\}dV + \int_{S_1}\{2t\mu_2\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} + t\lambda_2(\mathrm{tr}\boldsymbol{\varepsilon})^2 + 2\beta\mu_1\mathrm{tr}[\mathbf{u}\times(\mathbf{u}\nabla)\times\mathbf{n}]\}dS > 0, \quad (2)$$

where $\boldsymbol{\omega} = \mathrm{antisym}(\mathbf{u}\boldsymbol{\nabla})$ is the infinitesimal rotation tensor, $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} - \frac{1}{k}\mathbf{I}\,\mathrm{tr}(\boldsymbol{\varepsilon})$ is the deviatoric strain tensor, both associated with the perturbing displacement field $\mathbf{u}$; $\beta$ is an arbitrary parameter ($0 \leq \beta \leq 1$), introduced to permit varying the resulting inclusion moduli restrictions over what will turn out to be their full allowable range for composite stability, from positive definite ($\beta = 0$) to strongly elliptic ($\beta = 1$), so that our results will show how the required coating moduli for overall stability will vary as the restrictions on the inclusion moduli are varied within this range.

To derive Eq. (2), we employed the Kelvin identity $\boldsymbol{\varepsilon}:\boldsymbol{\varepsilon} = \boldsymbol{\omega}:\boldsymbol{\omega} + (\mathrm{tr}\boldsymbol{\varepsilon})^2 + \nabla\cdot[(\mathbf{u}\nabla)\cdot\mathbf{u} - \mathbf{u}(\nabla\cdot\mathbf{u})]$, the divergence theorem, $\mathbf{u}$ continuity, and the thin-coating assumption to approximate $\boldsymbol{\varepsilon}$ as independent of radius $r$ in the coating. Equation (2) is the key sufficient condition for stability for the two problems we analyze.

We first analyze the plane strain coated cylinder problem (explained above). In this case, Eq. (2) becomes, taking $k = 2$, and expressing the surface integral in terms of polar coordinates $r$, $\theta$, so $dS = ad\theta$:

$$\int_{A_1}\{2\mu_1[(1-\beta)\boldsymbol{\varepsilon}':\boldsymbol{\varepsilon}' + \beta\boldsymbol{\omega}:\boldsymbol{\omega}] + [\lambda_1 + (1+\beta)\mu_1](\mathrm{tr}\boldsymbol{\varepsilon})^2\}dA + \int_0^{2\pi}\{ta[2\mu_2(\varepsilon_{rr}^2 + \varepsilon_{\theta\theta}^2 + 2\varepsilon_{r\theta}^2) + \lambda_2(\varepsilon_{rr} + \varepsilon_{\theta\theta})^2] - 2\beta\mu_1[(u_r)^2 + (u_\theta)^2 + u_r u_{\theta,\theta} - u_\theta u_{r,\theta}]\}d\theta > 0, \quad (3)$$

per unit thickness, where $A_1$ is the in-plane area of $V_1$, a subscript comma denotes partial differentiation with respect to the subsequent subscript variable, and note that components of $\mathbf{u}$ and $\boldsymbol{\varepsilon}$ are evaluated on $r = a$ in the surface integral. We employ a Fourier series representation for $u_\theta$ on $r = a$, $u_\theta = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, $c_{-n} = \bar{c}_n$ (overbar is the complex conjugate), and no rigid-body motion requires $c_0 = c_1 = 0$. These and orthogonality of Fourier series show on $r = a$ that

$$\int_0^{2\pi}(u_\theta)^2 d\theta = 4\pi\sum_{n=2}^{\infty}\bar{c}_n c_n, \qquad \int_0^{2\pi}(u_{\theta,\theta})^2 d\theta = 4\pi\sum_{n=2}^{\infty}n^2\bar{c}_n c_n. \quad (4)$$

This shows on $r = a$ that

$$4\int_0^{2\pi}(u_\theta)^2 d\theta \leq \int_0^{2\pi}(u_{\theta,\theta})^2 d\theta. \quad (5)$$

Integrating the last term in Eq. (3) by parts, while requiring $\mathbf{u}$ to be continuous and single-valued, using Eq. (5), and recalling $\varepsilon_{\theta\theta} = (u_{\theta,\theta} + u_r)/r$, show Eq. (3) may be restated as

$$\int_{A_1}\{2\mu_1[(1-\beta)\boldsymbol{\varepsilon}':\boldsymbol{\varepsilon}'+\beta\boldsymbol{\omega}:\boldsymbol{\omega}]+[\lambda_1+(1+\beta)\mu_1](\mathrm{tr}\boldsymbol{\varepsilon})^2\}dA$$
$$+\int_0^{2\pi}\left\{ta[2\mu_2(\varepsilon_{rr}^2+\varepsilon_{\theta\theta}^2+2\varepsilon_{r\theta}^2)+\lambda_2(\varepsilon_{rr}+\varepsilon_{\theta\theta})^2]+2\beta\mu_1\left[\frac{3}{4}(u_{\theta,\theta})^2-a^2\varepsilon_{\theta\theta}^2\right]\right\}d\theta>0. \quad (6)$$

Since all terms involving the perturbing displacement field in Eq. (6) are now squared, sufficient conditions on the elastic moduli for stability under an arbitrary nonzero perturbing displacement field are that the coefficients of all different terms be positive. The following elastic moduli sufficiency requirements for stability may thus be directly read off from Eq. (6):

$$\mu_1>0, \quad \lambda_1>-(1+\beta)\mu_1, \quad \mu_2>\beta\frac{a}{t}\mu_1, \quad \lambda_2>0. \quad (7)$$

Weaker sufficient restrictions for stability are obtained by minimizing the integrand of the surface integral in Eq. (6) over all permissible $\mathbf{u}$ fields:

$$\mu_1>0, \quad \lambda_1>-(1+\beta)\mu_1, \quad \mu_2>\frac{\beta}{2}\frac{a}{t}\mu_1, \quad \lambda_2>-\frac{1-\beta\frac{a}{t}\frac{\mu_1}{\mu_2}}{1-\frac{\beta}{2}\frac{a}{t}\frac{\mu_1}{\mu_2}}\mu_2. \quad (8)$$

(Recall $0\le\beta\le1$.) Restrictions on Poisson's ratio $\nu$ in each material are immediate from Eq. (8) via the relation $2\nu/(1-2\nu)=\lambda/\mu$. Interpretations and implications of Eq. (8) are discussed following Eq. (11).

Next we analyze the three-dimensional coated sphere problem (explained above). Here, $k=3$ in Eq. (2), whose surface integral is expressed in spherical coordinates $r$, $\phi$, $\theta$, where $0\le\phi\le2\pi$, $0\le\theta\le\pi$, and $dS=a^2\sin\theta d\theta d\phi$. Using an approach similar to that in the plane strain case [employing integration by parts, $\mathbf{u}$ continuity, and requiring $\boldsymbol{\varepsilon}$ to be noninfinite at $\theta=0,\pi$], Eq. (2) becomes

$$\int_{V_1}\left\{2\mu_1[(1-\beta)\boldsymbol{\varepsilon}':\boldsymbol{\varepsilon}'+\beta\boldsymbol{\omega}:\boldsymbol{\omega}]+\left[\lambda_1+\frac{2}{3}(1+2\beta)\mu_1\right](\mathrm{tr}\boldsymbol{\varepsilon})^2\right\}dV$$
$$+\int_0^{2\pi}\int_0^{\pi}\left\{ta^2[2\mu_2(\varepsilon_{rr}^2+\varepsilon_{\theta\theta}^2+\varepsilon_{\phi\phi}^2+2\varepsilon_{r\theta}^2+2\varepsilon_{r\phi}^2+2\varepsilon_{\theta\phi}^2)+\lambda_2(\varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{\phi\phi})^2]\right.$$
$$\left.+2\beta a\mu_1\left[-a^2(\varepsilon_{\theta\theta}^2+\varepsilon_{\phi\phi}^2+2\varepsilon_{\theta\phi}^2)+(u_{\theta,\theta}-u_\theta\cot\theta)^2+4u_\phi^2+\frac{1}{2}\left(\frac{u_{\theta,\phi}}{\sin\theta}+u_{\phi,\theta}-3u_\phi\cot\theta\right)^2+\frac{u_{\phi,\phi}^2-4u_\phi^2}{\sin^2\theta}\right]\right\}\sin\theta d\theta d\phi>0. \quad (9)$$

The surface integral of the entire last bracketed quantity, excepting the strain terms, is nonnegative since that of the last fraction can be proved nonnegative by a Fourier series representation for $u_\phi$ as a function of $\phi$ at each $\theta$-value on $r=a$, then performing the $\phi$-integration, just like the logic leading to Eq. (5). Thus, from Eq. (9), we directly read off the following sufficient stability conditions on the elastic moduli for arbitrary nonzero perturbing displacement fields:

$$\mu_1>0, \quad \lambda_1>-\frac{2}{3}(1+2\beta)\mu_1, \quad \mu_2>\beta\frac{a}{t}\mu_1, \quad \lambda_2>0. \quad (10)$$

Minimization of the surface integral in Eq. (9) over all permissible $\boldsymbol{\varepsilon}$ fields shows the following weaker conditions are sufficient for stability:

$$\mu_1>0, \quad \lambda_1>-\frac{2}{3}(1+2\beta)\mu_1,$$
$$\mu_2>\beta\frac{a}{t}\mu_1, \quad \lambda_2>-\frac{2}{3}\mu_2\frac{1-\beta\frac{a}{t}\frac{\mu_1}{\mu_2}}{1-\frac{\beta}{3}\frac{a}{t}\frac{\mu_1}{\mu_2}}. \quad (11)$$

The restrictions on the elastic moduli for composite stability in the two-dimensional and three-dimensional cases, Eqs. (8) and (11), respectively, are very similar and have similar interpretations. The first two conditions of Eqs. (8) and (11) show that the weakest possible restrictions on the inclusion moduli arise from the choice $\beta=1$; these require the inclusion moduli to be strongly elliptic for overall stability. The second two conditions of Eqs. (8) and (11) with $\beta=1$ then show the coating moduli requirements for overall stability in this case. These coating moduli requirements are significantly stronger than positive definiteness (which would require $\mu_2>0$, $\lambda_2>-\mu_2$ in the 2D case, and $\mu_2>0$, $\lambda_2>-\frac{2}{3}\mu_2$ in the 3D case). Second, by varying $\beta$ within the range $0\le\beta\le1$, the first two conditions of Eqs. (8) and (11) show the full range of permissible inclusion moduli values, and their second two conditions show the associated required coating moduli values, for overall composite stability. To summarize: the results Eqs. (8) and (11) show that the two-dimensional coated cylinder, and the three-dimensional coated sphere composites will be stable under traction (dead load) boundary conditions so long as the inclusion moduli are at least strongly elliptic, and the coating moduli are sufficiently stiff; and if one considers $\beta$ to decrease from 1 to 0, the stability requirements on the inclusion moduli increase smoothly from strong ellipticity to posi-

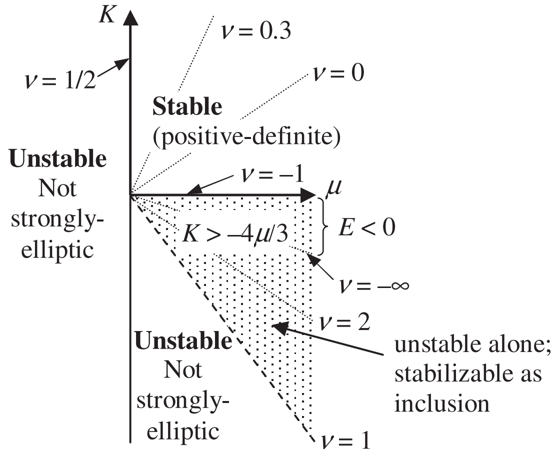


FIG. 2. Bulk $K$ versus shear $\mu$ modulus plot showing expanded stability regime (shaded) for inclusion material in 3D composites with a sufficiently stiff encapsulating phase [satisfying line 2 of Eqs. (11) or (12) with $\beta = 1$].

tive definiteness, while those of the coating moduli decrease smoothly from more restrictive than positive definiteness to positive definiteness, as expected.

For the three-dimensional coated sphere composite, Eqs. (11) imply the following restrictions on Poisson's ratio $\nu$ and the bulk modulus $K = \lambda + 2\mu/3$:

$$K_1 > -\frac{4}{3}\beta\mu_1, \qquad \frac{2\nu_1}{1-2\nu_1} > -\frac{2}{3}(1+2\beta),$$
$$K_2 > \frac{2}{3}\beta\frac{a}{t}\mu_1, \qquad \frac{2\nu_2}{1-2\nu_2} > -\frac{2}{3}\frac{1-\beta\frac{a}{t}\frac{\mu_1}{\mu_2}}{1-\frac{\beta}{3}\frac{a}{t}\frac{\mu_1}{\mu_2}}. \tag{12}$$

As noted, the least stable permissible inclusion can have merely *strongly elliptic* moduli; from Eqs. (11) and (12) with $\beta = 1$, this comprises the following restrictions on the isotropic elastic inclusion moduli:

$$\mu_1 > 0, \qquad \lambda_1 > -2\mu_1, \qquad -\infty < \nu_1 < \frac{1}{2},$$
$$1 < \nu_1 < \infty, \qquad K_1 > -\frac{4}{3}\mu_1, \qquad -\infty < E_1 < \infty, \tag{13}$$

where $E = 2\mu(1+\nu)$ is the Young or tensile modulus. Thus we have shown that the spherical composite will be stable for a substantial range of negative inclusion bulk modulus $K_1$ and for arbitrary values of the inclusion Young's modulus $E_1$, for a sufficiently stiff coating [one satisfying line 2 of Eqs. (11) or (12) with $\beta = 1$]. Figure 2 illustrates the expanded permissible regime of the inclusion moduli for overall composite stability.

Although the stability results proved here are for a thinly coated cylinder or sphere, they imply that the more general case of a body containing a negative-stiffness inclusion can also be stable, since adding positive-definite coating material will not decrease the stability of the overall composite [see Eq. (1)]. The requirements on the coating moduli for thicker coatings will almost certainly be less demanding than those derived here; an analysis of this case is underway.

Our analysis has treated the composite components as being in an unstrained ground state. The results are valuable theoretically in showing that the needed composite moduli restrictions are weaker than positive definiteness, and thus that the bounds on overall composite response can be evaded. In practice, some prestraining to produce a negative-stiffness inclusion response may be necessary. However, in many cases this will be very small (e.g., if a constrained ceramic phase-transforming inclusion is employed to produce the negative-stiffness response), and will further be limited by the necessarily stiff coating material. Thus the present analysis would seem to be a sensible first-order model of such practical situations.

The results proved here provide justification for the idea that dramatically improved overall composite material properties can be sought by permitting one phase to have negative stiffness, while retaining overall stability of the composite material. More broadly, our results suggest the strong possibility that materials containing an unstable phase of general type (not restricted to instability produced by negative stiffness) can be stabilized overall, and that therefore the search for novel and optimal overall material properties should be greatly broadened to include the possibly dramatic effects of controlled instability.

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