Spontaneous Emergence of Angular Momentum Josephson Oscillations in Coupled Annular Bose-Einstein Condensates

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We investigate the nonlinear dynamics of two coupled annular Bose-Einstein condensates (BECs). For certain values of the coupling strength the nonrotating state with uniform density is unstable with respect to fluctuations in the higher angular momentum modes. The Bogoliubov spectrum possesses two branches, one of which exhibits distinct regions of instability enabling one to selectively occupy certain angular momentum modes. For sufficiently long evolution times, angular momentum Josephson oscillations spontaneously appear, breaking the initial chiral symmetry of the BECs.

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One of the most famous paradigms of quantum physics is the existence of Josephson oscillations. They were first predicted for superconductors separated by an insulating layer [1]. Later, they have been observed in superfluid ^{3}He [2] and gaseous BECs [3,4]. Next to the oscillations of charge and/or particles between two modes, these systems can exhibit highly nonlinear dynamics with sometimes surprising behavior. In this Letter, we study one-dimensional (1D) BECs confined in two ring-shaped traps which are sufficiently close to each other to allow tunneling through the barrier between them. We demonstrate that the stationary state, in which only the zero angular mode is occupied by the BECs, becomes unstable for certain values of the coupling strength. For short propagation times, the angular momentum in each ring is conserved. For longer interaction times, however, angular momentum Josephson oscillations appear. This novel type of Josephson oscillation spontaneously breaks the initial chiral symmetry of the individual BECs. A spontaneously broken chiral symmetry in a two-dimensional spin 1 BEC of ^{87}Rb atoms has recently been explored in a work by Saito *et al.* [5] where the authors report a transfer of angular momentum from the spin to the spatial degrees of freedom leading to a spontaneous formation of vortices. This results from a dynamic instability which occurs at a specific ratio of the atom-atom interaction parameters of the $F = 0$ and $F = 1$ scattering channel. This ratio is fixed for a given atomic species. By contrast, in two coupled annuli, the instability leading to the chiral symmetry breaking occurs also in spin-polarized gases and is controllable by the tunnel coupling among the BECs. This, on the one hand, allows us to predict whether spontaneous symmetry breaking occurs at all. On the other hand, this constitutes a handle to selectively occupy specific angular momentum modes.

In the mean-field description, the evolution of a dilute gas of identical interacting bosons under the influence of the trapping potential $V(\mathbf{r})$ is governed by the Gross-Pitaevskii equation (GPE), which in cylindrical coordinates reads

$$i\hbar\partial_t\Psi = \left[\frac{\hbar^2}{2M}\left(-\partial_\rho^2 - \partial_z^2 + \frac{L_z^2}{\hbar^2\rho^2}\right) + V(\mathbf{r})\right]\Psi + g|\Psi|^2\Psi \tag{1}$$

where M is the atomic mass, g the nonlinear coupling constant, $\Psi = \Psi(\mathbf{r})$ the bosonic mean field, and $L_z = -i\hbar\partial_\phi$ the z-component of the angular momentum operator. The system consists of two BECs in parallel ring-shaped traps encircling the z-axis. The positions of the upper and lower rings are $\pm z_0$, respectively. Correspondingly, the trapping potential takes the form $V(\mathbf{r}) = V_\rho(\rho - \rho_0) + V_z(z, z_0)$. The first term provides harmonic radial confinement centered at $\rho = \rho_0$, and $V_z(z, z_0)$ creates a symmetric double well potential with its minima at $z = \pm z_0$. Both BECs reside in the radial ground state $\Psi_\rho(\rho)$ of $V_\rho(\rho - \rho_0)$. Vertically, they occupy the harmonic ground states $\Phi(z \pm z_0)$ which are localized in the upper and the lower well of $V_z(z, z_0)$, respectively. The total wave function of the system can then be written as $\Psi(\mathbf{r}) = \Psi_\rho(\rho) \times [\Phi(z - z_0)\chi_u(\phi) + \Phi(z + z_0)\chi_d(\phi)]$ where the indices u and d refer to the upper and lower rings. After inserting $\Psi(\mathbf{r})$ into the GPE, we obtain the two coupled equations

$$i\partial_\tau\chi_{u/d} = -\partial_\phi^2\chi_{u/d} + \kappa\chi_{d/u} + \gamma|\chi_{u/d}|^2\chi_{u/d}. \tag{2}$$

Here, we have introduced the scaled time $\tau = \frac{\hbar}{2MR^2}t$, the coupling $\kappa = -R^2\int dz\Phi(z+z_0)[\partial_z^2 - \frac{2M}{\hbar^2}V(z)]\Phi(z-z_0)$, and the interatomic interaction parameter $\gamma = \frac{2MR^2 g_{1D}}{\hbar^2} \times \int dz\Phi^4(z)$ with $R^{-2} = \int d\rho\rho^{-2}|\Psi_\rho(\rho)|^2$ [6]. We assume that the external confinement allows an effective 1D treatment of the BECs. The atom-atom interaction can then be described by the 1D coupling constant $g_{1D} = \frac{2\hbar^2}{M}\frac{a}{a_\rho^2}$ [7]. Here, a is the three-dimensional s-wave scattering length and a_ρ the harmonic oscillator length of the radial ground state. Equations, similar to Eq. (2) arise in the context of two coupled elongated condensates [8]. Ring traps additionally allow the existence of stationary currents.

In the angular momentum mode representation, the azimuthal wave function of the individual BECs can be written according to $\chi_{u/d} = (2\pi)^{-1/2}\exp(i\theta_{u/d}) \times \sum_m \alpha_m^{u/d}\exp(im\phi)$ with $\theta_{u/d}$ being the phase of the wave function in the respective annulus. The coefficients $\alpha_m^{u/d}$ are normalized such that $|\alpha_m^{u/d}|^2 = N_m^{u/d}$ corresponds to the number of particles residing in the m-th angular momentum mode. Hence, $\int d\phi|\chi_{u/d}|^2 = N^{u/d}$ corresponds to the total number of particles in each of the two annuli. Inserting the above expression for $\chi_{u/d}$ into Eq. (2), we find the system of coupled equations

$$i\partial_\tau \alpha_m^{u/d} = m^2\alpha_m^{u/d} + \kappa_{u/d}\alpha_m^{d/u} + \frac{\gamma}{2\pi}\sum_{nn'}\alpha_n^{u/d}\alpha_{n'}^{*u/d}\alpha_m^{u/d} \quad (3)$$

with $\kappa_{u/d} = \kappa^*_{d/u} = \kappa e^{i(\theta_d - \theta_u)}$. The first term represents the kinetic energy of the m-modes. The second term results from the tunnel coupling between the annuli. Because of their orthogonality only modes with the same m are coupled. Coupling between different m-modes is established by the third term which is due to the nonlinear mean-field interaction. We now seek a stationary solution of Eq. (3) for which in both of the annuli solely the $m = 0$ mode is occupied. This exists only for equal number of particles, i.e. $N_0^u = N_0^d = N_0$, and equal coupling $\kappa_u = \kappa_d = \kappa$. We then find the two solutions

$$\alpha_0^u = \pm\alpha_0^d = \sqrt{N_0}e^{i(\varepsilon\pm\kappa)\tau + i\theta}, \qquad \alpha_{m\neq 0}^{d/u} = 0 \quad (4)$$

with some arbitrary phase θ and $\varepsilon = \frac{\gamma N_0}{2\pi}$ being the nonlinear energy due to the interatomic interaction. Hence, the total two-dimensional wave function becomes either a symmetric or an antisymmetric superposition of the axial ground states of the two annuli: $\Psi_\pm(\mathbf{r},\tau) = \sqrt{N_0}e^{i\mu_\pm\tau + i\theta}\Psi_\rho(\rho)[\Phi(z+z_0) \pm \Phi(z-z_0)]$ with $\mu_\pm = \varepsilon \pm \kappa$ being the chemical potential.

In order to investigate the stability of these states with respect to fluctuations in modes with $m \neq 0$, we make the ansatz $\alpha_{m\neq 0}^{u/d} = e^{i\mu_\pm\tau}[u_m^{u/d}e^{-i\omega\tau} + v_m^{*u/d}e^{i\omega\tau}]$. Inserting this together with Eq. (4) into Eq. (3) yields after linearization in the u_m and v_m the eigenvalue equation

$$\begin{aligned} \omega u_m^{u/d} &= (m^2 + \mu_\pm)u_m^{u/d} + \varepsilon e^{i2\theta}v_{-m}^{u/d} + \kappa u_m^{d/u} \\ -\omega v_{-m}^{u/d} &= (m^2 + \mu_\pm)v_{-m}^{u/d} + \varepsilon e^{-i2\theta}u_m^{u/d} + \kappa v_{-m}^{d/u}. \end{aligned} \quad (5)$$

We then find the excitation spectrum consisting of the two branches

$$\omega_\pm = \sqrt{(m^2 + \varepsilon - \kappa \pm \kappa)^2 - \varepsilon^2}. \quad (6)$$

This result is obtained for either chemical potential $\mu_\pm$. The branch ω_+ corresponds to the well-known Bogoliubov spectrum [9] of a uniform BEC but with integer m. In this Letter, we consider only repulsive interatomic interaction, i.e., $\varepsilon \geq 0$. Thus, ω_+ is always a real number, and the coupled condensates are stable against fluctuations in this excitation branch. Note that ω_+ is independent of the coupling strength κ. The ω_- branch, on the other hand, depends on κ and results from the interaction among the coupled condensates. Its frequencies are either real or imaginary but never complex. For zero coupling, ω_- is identical with ω_+. If the nonlinear energy ε is zero, we find $\omega_+ = m^2$ and $\omega_- = m^2 - 2\kappa$. These frequencies belong to states which are symmetric or antisymmetric superpositions of m-modes of the upper and the lower annulus. The degeneracies of these states are lifted due to the coupling which results in an energy gap of 2κ.

We now turn to the case $\varepsilon \neq 0$. Figure 1 depicts for two values of the nonlinear energy ($\varepsilon = 1.0$ and $\varepsilon = 2.0$) the spectrum ω_- as a function of the coupling κ. The blue/solid curves represent the real part and the red/dashed curves the imaginary part of ω_-. First, let us consider relatively low energies ($\varepsilon < 2$) [Fig. 1(a)]. Starting from $\kappa = 0$, we notice that ω_- is positive and real and that it decreases monotonously to $\omega_- = 0$ at $\kappa = 1/2$. After this, one enters a region where ω_- is imaginary with $\mathrm{Im}(\omega_-) > 0$. The system is then unstable under fluctuations in the $m = \pm 1$ modes which grow at a rate of $\Gamma = 2\mathrm{Im}(\omega_-)$ [5,8]. As κ increases, regions of stability and instability follow one another. The latter are defined by $m^2/2 < \kappa < m^2/2 + \varepsilon$. For any given ε, the maximum growth rate $\Gamma_{\max} = 2\varepsilon$ is a universal quantity for all modes which is independent of m and is established at the coupling strengths $\kappa = \frac{1}{2}[m^2 + \varepsilon]$. Note that for sufficiently large ε, the unstable regions of two adjacent m-modes may even overlap, thus eliminating the stable region in between. This can be seen in Fig. 1(b), where for a nonlinear energy $\varepsilon = 2.0$, part of the regions of instability for the $m = 1$ and $m = 2$ modes overlap. Since the coupling κ is a function of the trapping potential, and as such experimentally accessible, the instability of the modes can be used to selectively affect one or more m-modes of the rings. For

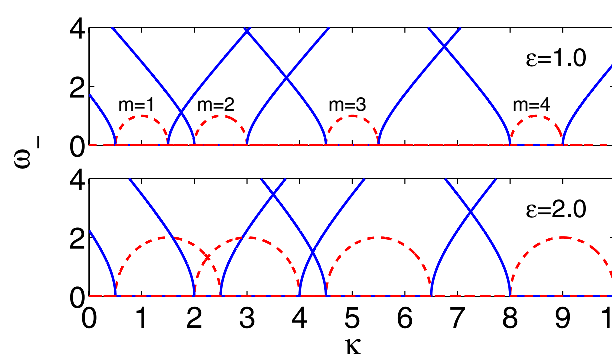

FIG. 1 (color online). Branch ω_- of the Bogoliubov spectrum of the coupled condensates plotted against the coupling strength κ for two values of the nonlinear energy ε (blue/solid: real part, red/dashed: imaginary part). In the top panel, the instable regions are labeled with the respective m number. Because of the symmetry of Eq. (5), the spectrum is symmetric with respect to a sign change of m.

example, at $\varepsilon = 1.0$, each mode is ''individually addressable'' through an appropriate choice of κ [see Fig. 1(a)]. At $\varepsilon = 2.0$ and $\kappa = 2.1$, on the other hand, both the $m = 1$ and $m = 2$ are unstable with respect to fluctuations.

The growth of the instable modes eventually leads to a break down of the linearized Eqs. (5) due to the interaction between higher lying m-modes. We therefore return to Eq. (3) and integrate them numerically. We choose $\alpha_0^{u/d} = \sqrt{N_0 + \delta_{u/d}}$ as an initial condition, i.e., an almost equal number of atoms in the $m = 0$ mode of both rings, while allowing for experimentally unavoidable particle number fluctuations of the order of $\delta_{u,d} = O(\sqrt{N_0})$. Adding these fluctuations has only minor influence on the numerical results. Following Saito *et al.* [5], we introduce a small seed in the lowest few angular momentum modes (up to $m = \pm 5$) with a magnitude of $10^{-4} \times \sqrt{N_0}$. Again, such a fluctuation is experimentally inevitable. We truncate the set of coupled Eq. (3) at the angular momentum mode $m = \pm 15$, well above the highest contributing mode. We verified the quality of the propagation by monitoring energy, norm, and angular momentum conservation. Since both annuli have a slight population difference, the nonlinear energy is calculated according to $\varepsilon = \frac{\gamma N_{\text{tot}}}{4\pi}$ with $N_{\text{tot}} = N_u + N_d$. An example of the numerical propagation can be seen in Fig. 2. We show the occupation N_m^u for $m = 0$, ± 1, ± 2 at the nonlinear energy $\varepsilon = 2.0$ [see also Fig. 1(b)] for two different coupling strengths κ. For $\kappa = 1.6$, only the $m = \pm 1$ modes are unstable. We observe for early times ($\tau < 10$) the predicted exponential increase in population with a rate of $\Gamma \approx 4$. At later times, $\tau > 9$ the population also in the $m = \pm 2$ modes increases slightly. This cannot be described by the linearized Eqs. (5). However, in the time window considered here, the population of the $m = \pm 1$ modes is more than 2 orders of magnitude larger than one of the $m = \pm 2$ modes. In the lower panel, we present the same plot for $\kappa = 3.2$. Here, we observe no population growth within the $m = \pm 1$ modes, but $N_{\pm 2}^u$ grows at a rate of $\Gamma = 3.9$. This clearly demonstrates the possibility of a selective angular momentum mode excitation by tuning κ.

Let us now turn to the angular momentum of the two BECs. The L_z expectation value of ω_- branch is $\langle L_z^u \rangle_-^m = \langle L_z^d \rangle_-^m = 0$ which implies an equal population of states with opposite m. Conversely, for modes of the branch ω_+, we find $\langle L_z^u \rangle_+ = \langle L_z^d \rangle_+ = \frac{m}{2} \frac{\sqrt{m^4 + 2\varepsilon m^2}}{m^2 + \varepsilon}$. However, according to Eq. (6), the value of ω_+ is always real, which results in a growth rate of $\Gamma = 0$. In Fig. 3, we present the angular momentum per particle and the relative particle number difference between the coupled BECs. The nonlinear energy is again $\varepsilon = 2.0$. The coupling is $\kappa = 2.1$, where both the $m = \pm 1$ and the $m = \pm 2$ are unstable. Until $\tau = \tau_{\text{osc}} \approx 11$, we find very small oscillations of the relative particle difference which are of the order of 10^{-2}. The decompositions into m modes show that this is due to small oscillations taking place between the $m = 0$ modes of the two rings (see also Fig. 2 for $\tau < 6$). These are ''ordinary'' Josephson oscillations. Here, no formation of currents, i.e., $\langle L_z^{u/d} \rangle \neq 0$, takes place in either of the annuli. For $\tau > 11$, the situation changes dramatically. We find particle oscillations up to $|N^d - N^u|/N_{\text{tot}} = 0.4$ and a nonzero expectation value for $\langle L_z^{u/d} \rangle$. The time τ_{osc} of the onset of this regime depends on the magnitude of the initial seed α_m. A rough estimate for τ_{osc} can be obtained from the equation $N_0 \approx N_m = |\alpha_m(t = 0)|^2 \exp(\Gamma \tau_{\text{osc}})$. These oscillations in $\langle L_z^{u/d} \rangle$ are due to the population of ω_+ modes caused by the nonlinear evolution of the system.

The oscillations spontaneously emerging for $\tau > \tau_{\text{osc}}$ are Josephson oscillations of the angular momentum. They break the chiral symmetry of the initial state's wave function where none of the rings carried a net angular momentum.

Finally, we turn to the experimental realizability of the ordinary and angular momentum Josephson oscillations. For ^{87}Rb and a ring radius of $\rho_0 = 1.2\ \mu$m, the energy scale given by the length of the ring evaluates to

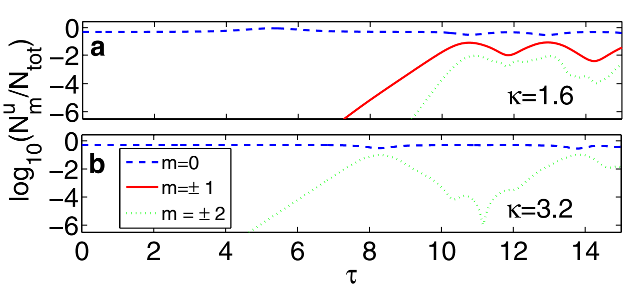

FIG. 2 (color online). Evolution of the occupation number N_m^u normalized to the total number of particles for the nonlinear energy $\varepsilon = 2.0$. For $\kappa = 1.6$ (panel a), we observe an exponential population increase in the $m = \pm 1$ modes. Population of the $m = \pm 2$ mode is also visible at later times but is suppressed by more than 2 orders of magnitude. For $\kappa = 3.2$ (panel b), the accumulation in population of the $m = \pm 1$ modes is suppressed, and only the exponential growth of the $m = \pm 2$ modes is visible.

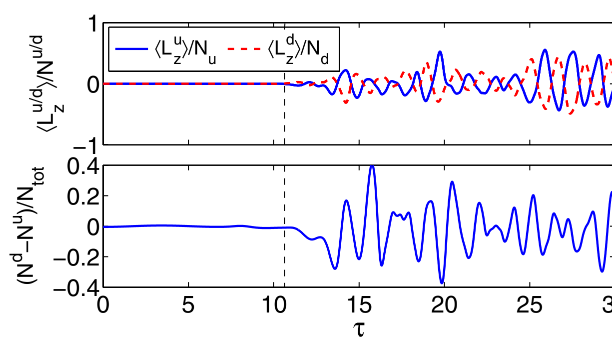

FIG. 3 (color online). Angular momentum per particle in units of $\hbar$ and relative particle difference between the two annuli ($\varepsilon = 2.0$ and $\kappa = 2.1$). For $\tau < 11$, small oscillations in the particle difference in the order of 10^{-2} take place. At $\tau = \tau_{\text{osc}} \approx 11$ (dashed vertical line), we observe the onset of oscillations with bigger amplitude which are accompanied by angular momentum Josephson oscillations.

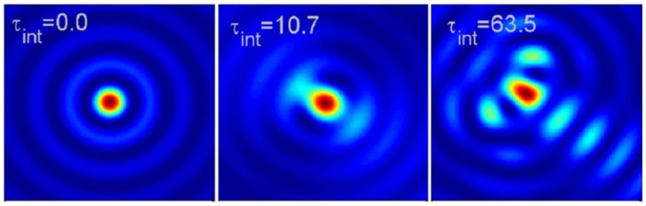

FIG. 4 (color online). Simulated TOF images for several interaction times. The parameters correspond to those of Fig. 2(a). The axes are in arbitrary units. For an interaction time of $\tau_{\text{int}} = 0.0$, only the $m = 0$ mode contributes to the momentum distribution yielding an image of the squared zeroth Bessel function. At $\tau_{\text{int}} = 10.7$, the $m = \pm 1$ are occupied as well which gives rise to an angular modulation proportional to $\sin^2(\phi)$ of the TOF image. For later times, several angular momentum modes are occupied. We show an example for $\tau_{\text{int}} = 63.5$. Here, both annuli carry a net angular momentum.

$E_0 \approx k_B \times 2$ nK, and consequently, we find a time scale of $\tau_0 \approx 4$ ms. For a radial oscillator length of $a_\rho = 0.3\ \mu$m, a nonlinear energy of $\varepsilon = 2.0$ is achieved for a particle number $N_0 \approx 50$ [10]. The experimental feasibility of building ring-shaped traps has been demonstrated recently [11,12], and further theoretical proposals for creating such a geometry exist [13,14]. We therefore hope the results presented in this Letter might stimulate further experiments.

The required smallness of the ring traps forbids direct *in situ* imaging of the BECs. Therefore, one has to employ time-of-flight (TOF) imaging. We consider an experiment where initially two annular BECs are created in two uncoupled ring traps from one single BEC. Subsequently, the barrier in z-direction is lowered such that a certain coupling strength κ is established. The system then evolves at constant κ for a certain time τ_{int} after which the trap is switched off. The TOF image is then taken after free expansion of the cloud. Here, we assume that only the wave function of one annulus is imaged; i.e., the atoms in the second annulus have to be removed [15]. The TOF method yields an image of the momentum distribution of the BEC [16–18]. This is equivalent to the squared modulus of the Fourier transform of the wave function $\Psi(\mathbf{r})$. Using the angular momentum mode decomposition α_m of the annulus which is to be probed and assuming that $a_\rho, a_z \ll R$, we obtain

$$\Psi(\mathbf{k}) \propto \sum_{m=\text{even}} (-1)^{m/2} J_{|m|}(kR)\alpha_m e^{im\zeta} - \sum_{m=\text{odd}} (-1)^{(m-1)/2} J_{|m|}(kR)\alpha_m e^{im\zeta}$$

with J_n being the n-th Bessel function of the first kind, $k = \sqrt{k_x^2 + k_y^2}$ and $\zeta = \arctan(k_y/k_x)$. Hence, imaging $|\Psi(\mathbf{k})|^2$ allows the reconstruction of the α_m for a small number of contributing modes (for an example see Fig. 4). This allows an experimental study of the instability regions simply by a analyzing TOF images.

In summary, in a system consisting of two ground state BECs in coupled rings, the occupation number of high angular momentum modes grows exponentially for well-defined coupling strengths. For small evolution times, a symmetric occupation of $\pm m$ modes takes place in each BEC accompanied by ordinary Josephson oscillations of the relative particle number. For later times, angular momentum Josephson oscillations spontaneously emerge. This novel type of Josephson oscillation breaks the initial chiral symmetry of the individual BECs.

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