Experimental Observation of Fermi-Pasta-Ulam Recurrence in a Nonlinear Feedback Ring System

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Fermi-Pasta-Ulam recurrence through soliton dynamics has been realized. The experiment used a magnetic film strip-based active feedback ring. At some ring gain level, a wide spin wave pulse is self-generated in the ring. As the pulse circulates, it separates into two envelop solitons with different speeds. When the fast soliton catches up and collides with the slow soliton, the initial wide pulse is perfectly reconstructed. The repetition of this process leads to periodic recurrences of the initial pulse.

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The unexpected recurrence of a nonlinear system back to an initial state was first discovered by Fermi, Pasta, and Ulam (FPU) in 1955 through the simulation of a one-dimensional lattice [1]. This FPU recurrence paradox was initially characterized by Fermi as a ‘‘little discovery.’’ This discovery, in fact, marked a true sea change in modern science [2]. On the one hand, it ushered in the age of computational science through the introduction of computer simulation for the first time. At the same time, it marked the birth of nonlinear science. It led to both the discovery of solitons and the widespread awareness of deterministic chaos.

The FPU recurrence paradox remained a complete mystery until Zabusky and Kruskal (ZK) discovered solitons in 1965 [3]. In an attempt to solve the FPU paradox, Zabusky and Kruskal reduced the FPU problem to the Korteweg–de Vries (KdV) equation based on a so-called continuum approximation and numerically discovered solitons. In terms of soliton dynamics, they explained that with time, a large-amplitude periodic wave described by the KdV equation can break up into a family of solitons with different speeds. When the fast solitons catch up and collide with the slow solitons, there is a reconstruction of the initial periodic wave.

The ZK 1965 work did even more than provide the first solution to the FPU paradox and mark the birth of soliton science. It also showed, at least theoretically, that one can actually realize FPU recurrence through the excitation of a large-amplitude periodic wave in a soliton-supporting nonlinear system. Attempts to realize such a FPU recurrence experimentally have been made for periodic waves in electrical networks [4], plasmas [5], and magnetic films [6]. Here, the breakup of the initial pulses into solitons and the subsequent overtake and collision of these solitons were observed. Bona fide recurrence, however, was not observed. First, the recurrence to the initial state was not exact because of energy decay. In addition, the energy decay also precluded the realization of more than a single recurrence. It is to be emphasized that the main reason for these failures lies in the dissipation present in the systems.

This Letter reports on the realization of an exact and periodic FPU type of recurrence. This has been achieved through the novel use of feedback to circumvent the dissipation problem. Specifically, one starts with a nonlinear pulse in a soliton-supporting one-dimensional medium. One then feeds the amplified output signal back to the input to produce a soliton-supporting ‘‘conservative’’ nonlinear ring system. Here, the conservative does not mean that the ring system is free of dissipation. Rather, it means that the wave dissipation is compensated by the active feedback. A circulating pulse in such a ring is topologically equivalent to a periodic wave train in a one-dimensional conservative system as studied in the ZK work [3]. As such, the nonlinear pulse experiment in such a ring is expected to show a true dissipation free recurrence response.

The experiment used a magnetic film strip–based active feedback ring. The magnetic film strip served as a nonlinear dispersive medium for the propagation of spin waves [7,8]. The propagation geometry was chosen to give an attractive or self-focusing nonlinearity that supports the formation of bright spin wave envelope solitons [9]. The active feedback allowed for the self-generation of a wide spin wave pulse. As the pulse circulates in the ring, it separates into two envelope solitons with different speeds. After many circulations, the fast soliton catches up and collides with the slow soliton and the initial wide pulse is perfectly reconstructed. The repetition of this soliton process leads to periodic recurrences of the initial pulse. Remarkably, the splitting of the initial pulse into separate solitons and the recombination of these solitons to reform the initial pulse over and over again is in nearly perfect agreement with the ZK prediction.

Figure 1 shows the experimental setup. The magnetic yttrium iron garnet (YIG) film strip is magnetized to saturation by a static magnetic field parallel to the length of the strip. This configuration allows for the propagation of backward volume spin waves [7–9] and the formation of bright spin wave envelope solitons [9]. Two microstrip transducers are placed over the YIG strip for the excitation

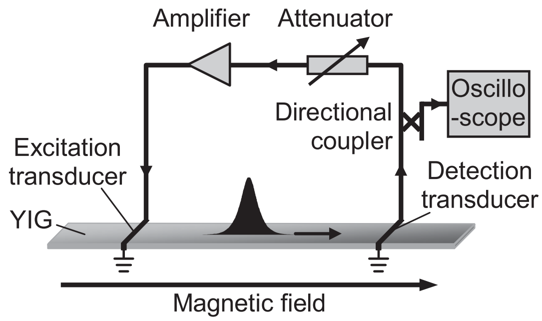

FIG. 1. Diagram of YIG strip-based active feedback ring system.

and detection of spin waves. The detection transducer is connected to the excitation transducer through a linear broadband microwave amplifier to form an active feedback ring. The ring gain is controlled through an adjustable microwave attenuator. The ring signal is sampled through a directional coupler and is analyzed with a broadband microwave oscilloscope. For the data shown below, the YIG film strip was 10.8 mm thick, 2 mm wide, and 37 mm long. The magnetic field was held at 970 Oe. The microstrip transducers were 50 mm wide and 2 mm long. The separation between the transducers was set to be 5.7 mm.

The feedback ring system can have a number of resonance eigenmodes that exhibit low decay rates. For a magnetic film feedback ring, the eigenmode frequencies are determined by the phase condition $k(\omega)l + \phi_0 = 2\pi n$ where k is the spin wave wave number, ω is the spin wave frequency, l is the transducer separation, ϕ_0 is the phase shift introduced by the feedback circuit, and n is an integer. The eigenmode frequencies and their spacing can be adjusted through a change in the $k(\omega)$ dispersion function and/or the transducer separation l. The dispersion function, in turn, can be controlled through the film parameters and the magnetic field [7]. At a low ring gain G, all of these eigenmodes experience an overall net loss and there is no spontaneous signal in the ring. If the ring gain is increased to a certain threshold level, here taken as $G = 0$, the eigenmode with the lowest decay rate will start to self-generate and one will obtain a continuous wave response at this eigenmode frequency. A further increase in the ring gain results in the generation of additional modes through a four-wave process. In the time domain, this corresponds to the formation of a spin wave pulse that circulates in the ring. The circulation period is given by the sum of the spin wave propagation time l/v_g in the film, where v_g is the group velocity, and the signal propagation time t_0 in the feedback circuit. Typically, the time l/v_g is on the order of 100 ns, while the time t_0 amounts to a few nanoseconds at most [10,11].

The power of the circulating spin wave pulse increases with the ring gain. At some threshold power for which the nonlinearity is strong enough for the nonlinearity-induced pulse narrowing to balance the dispersion-induced pulse broadening, the pulse evolves into an envelope soliton [12,13]. With a further increase in the ring gain, the pulse power becomes too high to maintain a single soliton state and the pulse breaks up into two solitons with different speeds. The slow overtake and subsequent collision of these two solitons produce the FPU recurrence that is demonstrated below.

Figure 2 shows output signals for three different ring gain levels. Graphs (a), (b), and (c) show power versus time profiles measured at $G = 0.2$, 0.3, and 0.6 dB, respectively. For easy comparison, the three signal traces are shown with the same power and time scales. Graph (d) shows expanded displays of the two pulses at 60 ns and 2920 ns from (c). Graph (e) shows the carrier waves for the two pulses in (d). The curve shows a sine function fit to the data.

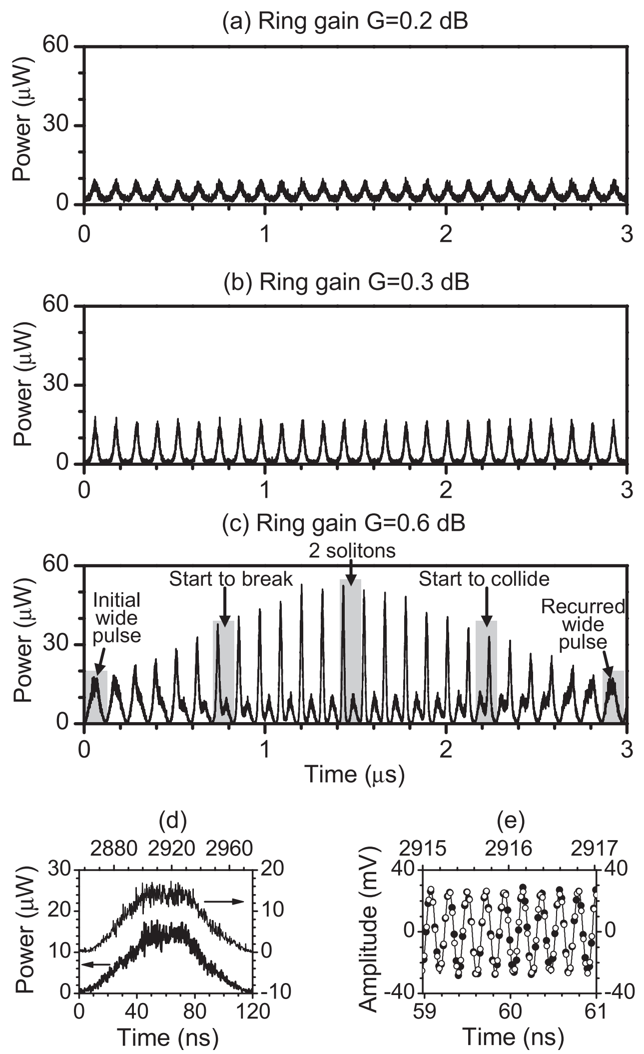

FIG. 2. Graphs (a), (b), and (c) show the power versus time profiles measured at ring gains of $G = 0.2$ dB, 0.3 dB, and 0.6 dB, respectively. Graph (d) shows the two pulses at 60 and 2920 ns from (c) on expanded scales. Graph (e) shows the carrier waves for the two pulses in (d). The curve shows a sine function fit.

The data in Fig. 2 demonstrate the realization of the FPU recurrence. In (a), one has a uniform pulse train that corresponds to a stable circulation of a single spin wave pulse in the ring. With an increase in the ring gain to 0.3 dB, as in (b), the amplitude of the pulse increases while the width of the pulse decreases. Indeed, the half-power width decreases from about 33 ns for $G = 0.2$ dB to about 24 ns for $G = 0.3$ dB. Such a self-narrowing effect results from attractive nonlinearity, and the circulating pulse now corresponds to a spin wave envelope soliton. The soliton nature of this pulse is confirmed by the hyperbolic secant shape [12,13] and a constant phase profile [14].

With a further increase in the ring gain to 0.6 dB, the pulse loses its soliton nature and evolves into a wide pulse. This corresponds to the leftmost pulse in (c). Its amplitude is about the same as for the soliton in (b). Its width of about 60 ns, however, is significantly larger than the width of the soliton in (b). This wide pulse is not stable [15,16], and it gradually breaks into two solitons after several round trips. This breakup is evident from the left half of the trace in (c). Note that the leading soliton is taller than the initial wide pulse, while the trailing soliton is shorter. Here too, the soliton nature of these pulses is evident from their hyperbolic secant shapes and constant phase profiles, as well as additional soliton signatures considered below. One critical point is that these solitons have amplitude-dependent speeds; the tall soliton travels faster than the short soliton. Because of this property, the tall soliton catches up and collides with the short soliton after several round trips. This is evident from the right part of the trace in (c). This process leads, in turn, to a perfect recurrence through a matchup of the left and right most pulses in (c). This perfect matchup is made even clearer from the superimposed responses in (d). The carriers also match. This is evident in (e) where both carrier waves are nicely fitted by one and the same sine function.

These results provide a perfect demonstration of FPU recurrence. One key for such a realization lies in the amplitude-dependent speed property of the solitons, just as predicted by Zabusky and Kruskal [3]. It is this amplitude-dependent speed that makes the slow overtake possible. A second key element of the process is the active nature of the ring. Without the amplified feedback, pulse decay would dominate the response, and the recurred pulse, if any, would have a much lower amplitude. It is this ring feature that makes the perfect recurrence possible.

Since solitons can survive collisions with other solitons [3,12], one would naturally expect that the repetition of the soliton collision process described above and the periodic recurrences to the initial pulse could occur for a very long time. Figure 3 shows that such an extended recurrence can actually be realized. Graph (a) shows the ring signal over a relatively long period of 10 μs. The experimental conditions are the same as for the data in Fig. 2(c). Graphs (b) and (c) show plots of the corresponding tall and short soliton peak times versus the number of round trips, respectively. The circles, triangles, and squares show the data for the solitons before the first collision, after the first collision, and after the second collision, respectively. The solid, dashed, and dotted lines are linear fits to the circle, triangle, and square data, respectively. The insets show selected segments of the data and the fits on expanded scales, as indicated.

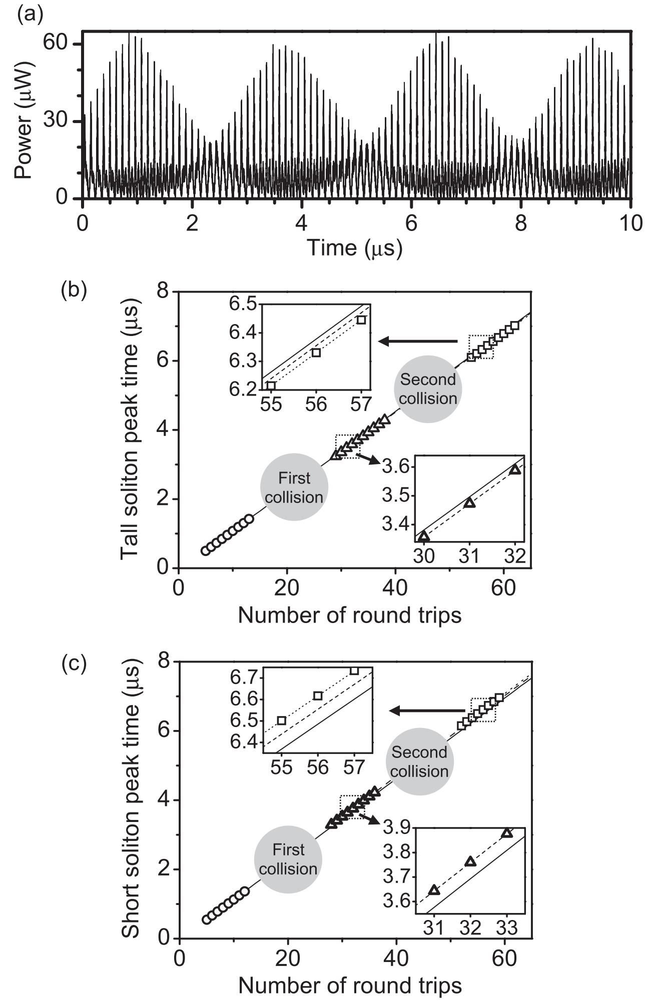

FIG. 3. Graph (a) shows the ring signal measured at a ring gain of 0.6 dB. Graphs (b) and (c) show the tall and short soliton peak times versus the number of round trips, respectively. The circles, triangles, and squares are for the solitons before the first collision, after the first collision, and after the second collision, respectively. The solid, dash, and dot lines show linear fits to the circle, triangle, and square data, respectively.

The data in Fig. 3(a) show the perfect periodic nature of the recurrence. Because response is periodic, the definition of the initial state is somewhat arbitrary. No matter which initial state is chosen, however, one always sees multiple recurrences to the initial state. As an example, if one takes the pulse at 0 ns as an initial state, one observes three full recurrence periods. Such an extended periodic response is the signature of FPU recurrence. As above, the key to this realization lies in the active feedback. It is believed that the recurrences to the initial pulse are exact. Any small differences between the initial and the recurred pulses may be attributed to the fact that the collisions do not always occur at the exact same positions, while the signals are always recorded at one and the same position.

Figure 3 also shows a number of basic soliton features. First, graph (a) demonstrates the ability of the solitons to survive collisions. It is this collision-survival property that allows for the repetitive collisions and the multiple recurrences observed in the first place. Second, graphs (b) and (c) show that the collisions also produce phase shifts in the peak time versus round trip number trajectories for a given soliton. For the tall soliton in (b), a collision causes the peak time to advance. For the short soliton in (c), in contrast, a collision causes a delay. Without these phase shifts, the recurrence period would be longer than that shown in (a). One can also see that the trajectories in (b) and (c) are perfectly straight. This demonstrates another key feature of solitons, namely, that they travel with a constant speed. In addition, one can see that all of the fitting lines in the insets are parallel to each other. This means that the solitons always recover their original speeds after collision.

The above results demonstrate an exact and periodic FPU recurrence. The results also provide direct experimental evidence for the ZK FPU recurrence interpretation in terms of soliton dynamics. Recall that the ZK interpretation was specifically for the class of nonlinear systems described by the KdV equation [3]. The YIG film based nonlinear ring here, however, does not constitute a KdV system. In this sense, the results also provide direct evidence for the universality of the ZK interpretation for a general nonlinear system. It is important to note that the present recurrence is realized through the overtaking collision of two solitons, while the recurrence dynamics in the ZK model involves as many as eight solitons. In other words, the present recurrence represents a simple but revealing case of the ZK interpretation. The experimental realization of the more aesthetic FPU recurrence that takes place through multiple collisions between a large number of solitons remains as a fascinating and challenging subject in nonlinear science.

In line with the above, the present results demonstrate that one can realize FPU recurrence in any practical soliton-supporting systems as long as the dissipation is compensated. This practical realization of FPU recurrence may well lead to novel methods for secure communications, among other application. One could, for example, (i) code information into the solitons, (ii) use a nonlinear ring system [11,17,18] to combine these solitons into a single pulse, (iii) transfer the combined pulse in a linear communication channel, and then (iv) use a secondary but identical nonlinear ring system to provide for the breakup of the pulse into solitons. One could code an information signal into the number and/or order of solitons. One could also use the soliton modulation schemes proposed by Suzuki *et al.* [19,20].

It is useful to note that the ZK interpretation is not the only solution to the FPU recurrence paradox [21,22]. The FPU paradox could also be interpreted, for example, in terms of modulational instability (MI) [22,23]. Indeed, MI-associated recurrence has also been experimentally observed in deep water [24] and optical fibers [25].

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