# Formation and Evolution of Structure in Loop Cosmology

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Inhomogeneous cosmological perturbation equations are derived in loop quantum gravity, taking into account corrections, in particular, in gravitational parts. This provides a framework for calculating the evolution of modes in structure formation scenarios related to inflationary or bouncing models. Applications here are corrections to the Newton potential and to the evolution of large scale modes which imply nonconservation of curvature perturbations possibly noticeable in a running spectral index. These effects are sensitive to quantization procedures and test the characteristic behavior of correction terms derived from quantum gravity.

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Cosmology has provided a successful paradigm for structure formation in our Universe through an inflationary phase [1] in early stages. Conceptually, however, the scenario is incomplete due to the presence of past singularities [2]. At such a singularity, the classical theory of general relativity breaks down and has to be replaced by an extended framework which remains well defined even at very high curvatures. Since this requires modifications to general relativity at early stages of cosmic evolution, there can then also be corrections to the usual scenario of structure formation which might eventually be observable. While dimensional arguments and low energy effective theory indicate that effects are very small, given by the tiny ratio of the Planck length $\ell_P = \sqrt{G\hbar}$ to the Hubble length $H^{-1}$, a detailed analysis is required and may reveal more sizeable effects. This is what we provide in this Letter in the framework of loop quantum gravity [3], a nonperturbative background independent approach to quantize gravity.

Loop quantum gravity is one of the approaches where singularity resolution has been investigated using loop quantum cosmology [4] which results in the resolution of singularities in various situations including inhomogeneous ones [5–8]. Semiclassical bounce pictures in special models have been described in [9–12]. A key role is played by the underlying quantum nature of spatial geometry [13]. With such a discrete structure underlying classical space-time, effects not captured by low energy effective theory become possible. In particular, there are large dimensionless parameters, such as the number of spatial lattice sites in a discrete state, which can always spoil dimensional arguments. In such a context, orders of magnitude of quantum corrections can only be estimated with a detailed analysis of the effective equations arising from quantum gravity. Suitable techniques going beyond low energy effective theory are now available and are applied here.

On larger scales farther away from the classical singularity one can use effective equations for the behavior of inhomogeneous perturbations, i.e., equations which are of classical type but amended by quantum correction terms as known from effective actions. Such equations can be used to test the semiclassical viability of crucial ingredients of quantum gravity. At the same time, they allow a detailed study of the formation and evolution of structure including quantum corrections.

In earlier papers [14,15], candidate effective equations have been used for inhomogeneous scalar fields on an isotropic metric background. This allowed preliminary indications, but no systematic derivation or reliable predictions. It therefore remained unclear which effects are to be expected. There are, for instance, cancellations in metric components on an isotropic background which hide terms that may or may not be modified by quantum gravity effects. Also, only corrections in the nongravitational part of the field equations were considered, while the more complicated gravitational part plays a crucial role, too. Ignoring such corrections does not only affect details but may even change aspects such as the scale invariance of the expected perturbation spectrum which is already highly constrained observationally.

For a reliable evaluation of this framework it is therefore essential to derive full perturbation equations which take into account corrections of gravitational dynamics. This is reported here for scalar metric perturbations, which in Newtonian gauge are of the diagonal form

$$\delta ds^2 = -2a(\eta)^2\Phi(\eta, x)(d\eta^2 + \delta_{ab}dx^a dx^b) \qquad (1)$$

on a flat background and in conformal time $\eta$. Other gauges and modes can be included similarly, although we do not do this here for the sake of simplicity. Although quantum gravity is not formulated for classical metrics, the form of the metric (1) plays the role of selecting the corresponding quantum regime where an effective description is derived. This happens by picking semiclassical states of quantum gravity which are peaked on the given class of metrics; i.e., expectation values of metric operators are of the prescribed form and fluctuations around these values are small. Expectation values of the Hamiltonian operator in those states give the Hamiltonian of the effective theory and thus effective equations [16,17]. Although the full quantization is background independent and nonperturbative, which is crucial for some properties of the quantum theory such as its spatial discreteness, a cosmological background is introduced in evaluating the theory through states and effective equations. This puts the scenario in the usual context of cosmological perturbation theory, albeit including quantum corrections. Being derived from general semiclassical states which are not Lorentz invariant unless one restricts oneself to a vacuum state, effective equations may not be manifestly covariant even if the underlying quantum theory is covariant [17].

Compared to the standard derivation of cosmological perturbation equations [18], loop quantum gravity is in a different situation because it is based on a canonical quantization. Lagrangians are thus not quantized directly, but Hamiltonians are used, which provides an alternative but fully equivalent classical formulation. At this step, no new physics enters but it makes the formulation of quantum gravity possible. To apply this to the question of interest here requires a derivation of perturbation equations in a canonical framework, which allows one to include effective quantum modifications which arise for the effective Hamiltonians. Correction terms in the evolution equations then follow uniquely, but in an indirect manner. Thus, basic quantum modifications can have several complicated effects in the perturbation equations, which allow one to test the underlying theory in a nontrivial manner.

For loop quantum gravity, in particular, quantum Hamiltonians are lattice operators taking into account the spatial discreteness of quantum geometry [19,20]. States are supported on lattices in space, which we assume here to be regular to simplify calculations; otherwise some coefficients can change but not by orders of magnitude. Lattice links are labeled by quantum numbers $p_{v,I}$ corresponding to elementary areas (centered at a vertex $v$ with transversal direction $I$) building up space. As seen from our final equations, the lattice does not introduce a preferred direction because the quantum Hamiltonian acting on the corresponding state is direction independent. A Hamiltonian is constructed from basic operators which are the elementary areas with eigenvalues $p_{v,I}$ and shift operators in those labels (related to curvature) [21]. Variables are thus discrete, associated with lattices, and a classical geometry arises only in a continuum limit. The size of the $p_{v,I}$ is given by the state as a multiple of $\ell_P^2$, which is a general parameter constructed from $G$ and $\hbar$ without input from quantum gravity. Minimum values where nonperturbative quantum effects are significant are $p_{v,I} \approx \ell_P^2$, but actual values of $p_{v,I}$ in a semiclassical state can well be larger giving rise to smaller quantum effects of the order $\ell_P^2/p_{v,I}$. These are the terms whose cosmological implications we will study here.

From lattice operators one first obtains an effective Hamiltonian, by taking an expectation value in a semiclassical state, as a function of the lattice areas rather than of the spatial metric. Coming from a lattice, such a function does not coincide with the classical one but contains discretization and other effects (whose magnitudes are close to the extrinsic curvature scale) in addition to $\ell_P^2/p_{v,I}$ terms. Since $p_{v,I}$ refers to the lattice size rather than the Hubble area, corrections can be much larger than $\ell_P^2 H^2$ as it was expected without discreteness. For $p_{v,I} \approx \ell_P^2$ even nonperturbative quantum effects have to be included, but for a semiclassical geometry this cannot arise. On the other hand, those corrections become arbitrarily small for $p_{v,I} \to \infty$, but there is an upper limit for $p_{v,I}$ because large $p_{v,I}$ imply large lattice sites. On length scales of $\sqrt{p_{v,I}}$, discreteness is noticeable which must thus be much smaller than scales probed by particle physics. Thus, a conservative upper bound is $\ell_P/\sqrt{p_{v,I}} \gg 10^{-15}$. But during inflation energy densities are much higher, up to $G\rho \approx 10^{-6}$ in Planck units, which requires $\ell_P^2/p_{v,I} \gg 10^{-6}$. This also means that extrinsic curvature terms given through the Friedmann equation by $\sqrt{Gp_{v,I}\rho}$ and thus higher curvature corrections are small. Although precise estimates of correction terms require detailed constructions of semiclassical states, the interplay of different corrections already suffices for a rough estimate of orders of magnitude.

For regular lattices, all types of corrections can be determined explicitly for a given Hamiltonian [22]. The Hamiltonian itself, however, is not fixed uniquely so far but subject to ambiguities such as the ordering of operators which do not commute in a quantum theory. A choice of Hamiltonian, including several ambiguity parameters, corresponds to a fixed theory which can be tested phenomenologically. One can test precise aspects to constrain such parameters, or consider the whole class of possibilities allowed in the framework of loop quantum gravity and check whether this general behavior is viable at all. At the current stage, the second possibility is more reasonable to pursue and already very instructive due to tight constraints on the general properties of operators. This gives rise to the indicated quantum corrections when the resulting effective equations are expanded: First, spatial discreteness implies the replacement of differential by difference operators which, when expanded semiclassically on a background, result in higher derivative and higher curvature corrections. Second, inverse powers of metric components occur in Hamiltonians which would classically diverge near a singularity but are modified at small scales by quantum effects [23]. The latter corrections of perturbative form $\ell_P^2/p_{v,I}$ are most relevant for sub-Planckian curvature which is our present focus.

Correction functions in coefficients of the effective Hamiltonian thus depend on the lattice areas $p_{v,I}$ rather than a continuous field such as the spatial metric. Moreover, they depend on time only implicitly through the time dependence of $p_{v,I}$. Qualitatively, such a correction function $\alpha(p_{v,I})$ behaves in a way which approaches classical behavior $\alpha = 1$ for large $p_{v,I}$ but leads to suppressions of otherwise diverging inverse powers for small $p_{v,I}$ [24]. Most important for us is that any correction function increases for very small $p_{v,I}$, reaches a peak of height larger than 1, and then approaches the classical expectation $\alpha = 1$ from above in a perturbative expansion in $\ell_P^2/p_{v,I}$. For perturbations around an isotropic geometry, one can express these corrections as functions of $H$ since $p_{v,I}^{-1} \approx \mathcal{N}^{2/3}H^2$ for $\mathcal{N}$ lattice sites of volume $p_{v,I}^{3/2}$ in a Hubble volume $H^{-3}$. The large factor $\mathcal{N}^{2/3}$ thus magnifies all corrections $\ell_P^2 H^2$ expected in low energy effective theory. From the perturbative metric (1) it follows, on the other hand, that $\mathcal{N}^{2/3}p_{v,I}(\eta) \propto a(\eta)^2[1 - 2\Phi(\eta, v)]$, which allows one to write all effective equations in terms of the scalar perturbation $\Phi$.

The relevant gravitational dynamics is determined by the Hamiltonian [25,26]

$$\int d^3x N \epsilon_{ijk} \frac{[2\partial_a \Gamma_b^i + \epsilon_{ilm}(\Gamma_a^l \Gamma_b^m - K_a^l K_b^m)]E_j^a E_k^b}{\sqrt{|\det E|}}, \quad (2)$$

expressed in basic fields $(E_i^a, K_b^j)$ which occur in loop quantum gravity. Here, $E_i^a$ is related to the spatial metric $q_{ab}$ by $E_i^a E_i^b = q^{ab} \det q$ and $K_a^i$ is the canonical momentum of $E_i^a$ (related to extrinsic curvature). The connection $\Gamma_a^i$ depends on spatial derivatives of $E_i^a$ and its inverse. The lapse function $N$ is a free function but will be specified when choosing a gauge.

For the effective Hamiltonian we keep only corrections for inverse powers of metric components, disregarding higher curvature corrections as it is adequate for sub-Planckian densities. There are two contributions by inverse powers of the fields, the one explicit in (2) and the other in connection components. We thus have two correction functions, $\alpha$ multiplying the whole integrand and $\beta$ multiplying connection components. They appear in different terms and will play quite different roles. There are thus classes of correction functions whose structure is determined theoretically and whose parameters, describing their precise shape, can be restricted by observations or other means such as internal consistency. Despite the nonuniqueness in parameters, crucial modifications are thus characteristic in the general form.

As in any Hamiltonian system, the Hamiltonian generates equations of motion. For the corrected Hamiltonian with $N = a(1 - \Phi)$, corrected perturbation equations of scalar modes in conformal time $\eta$ take the form [25]

$$\alpha^2\beta\nabla^2\Phi - 3\mathcal{H}\dot{\Phi} - 3\left(1 - \frac{\alpha'\bar{p}}{\alpha}\right)\mathcal{H}^2\Phi = -\frac{\kappa}{2}\alpha\bar{p}\delta T_0^0, \quad (3)$$

$$\ddot{\Phi} + 2\Phi\dot{\mathcal{H}}\left(1 - \frac{\alpha'\bar{p}}{\alpha}\right) + 3\dot{\Phi}\mathcal{H}\left(1 - \frac{2}{3}\frac{\alpha'\bar{p}}{\alpha}\right) + \frac{\alpha\beta}{3}\nabla^2\Phi[\alpha(\beta - 1) - 4\alpha'\bar{p}] + \Phi\mathcal{H}^2\left[1 - 5\frac{\alpha'\bar{p}}{\alpha} + 4\left(\frac{\alpha'\bar{p}}{\alpha}\right)^2 - 2\frac{\alpha''\bar{p}^2}{\alpha}\right] = \frac{\kappa}{2}\alpha\bar{p}\delta T_a^a, \quad (4)$$

$$\partial_a[\dot{\Phi} + \mathcal{H}\Phi(1 - 2\alpha'\bar{p}/\alpha)] = -\frac{\kappa}{2}\bar{p}\delta T_a^0. \quad (5)$$

A prime denotes derivatives with respect to $\bar{p} = a^2$, which is the sum of all $p_{v,I}$, $\mathcal{H} = \dot{a}/a$, $\kappa = 8\pi G$, and $\delta T_b^a$ are perturbations of the stress-energy tensor. For classical values of the correction functions, $\alpha = \beta = 1$, we obtain the classical perturbation equations [18], which demonstrates the correct classical limit on very large scales of the effective theory. On intermediate scales, however, there are corrections which may lead to detailed and reliable viability test of cosmological scenarios in loop quantum gravity and in proposals for potentially observable effects.

Our effective equations include gravitational corrections from quantum gravity directly related to its basic discrete structures. They suggest several applications on different scales, perhaps allowing tests of different regimes of quantum gravity. First, we isolate gravitational effects by assuming an effective perfect fluid background such that $T_{aa} = \bar{P} + \delta P = wT_{00} = w(\bar{\rho} + \delta\rho)$ with a constant $w$ and $T_{0a} = (\bar{\rho} + \bar{P})u_a$ with the energy density $\rho$, pressure $P$, and velocity $u_a$. For curl-free velocity $u_a = \partial_a u$, we combine (3) and (5) to a Poisson equation

$$\nabla^2\Phi - \delta\mu(\bar{p})^2\Phi = \frac{\kappa\bar{p}}{2\alpha\beta}[\delta\rho + 3\alpha^{-1}\mathcal{H}(\bar{\rho} + \bar{P})u], \quad (6)$$

with $\delta\mu(\bar{p})^2 = 3\mathcal{H}^2\alpha'\bar{p}/\alpha^3\beta$. For classical values, $\delta\mu = 0$ and we obtain the general relativistic Poisson equation corrected only by a pressure term. The correct classical limit is thus obtained as $\alpha \to 1$ for $\ell_P \to 0$. But since $\ell_P$ is nonzero, quantum effects always remain: to leading order we derive Newton's potential which receives quantum corrections. Our derivation of the Newton potential is alternative to that proposed in [27] and conceptually quite different. The precise value of corrections depends on the Hubble parameter, or the cosmological constant. This refers only to perturbations around a flat isotropic cosmology as this is the setting in which we derived our equations. The result on the Newton potential thus does not directly apply to the solar system for which perturbation equations around the Schwarzschild solution are required. They can be derived by the same methods which are, however, technically more involved for a curved and inhomogeneous background. From the general procedure we expect that the Hubble parameter occurring in correction terms will effectively be replaced by the solar mass.

Additional applications arise for structure formation. For the main effect we combine Eqs. (3) and (4) to eliminate stress energy. Still, for an effective perfect fluid,

$$\ddot{\Phi} + 3(1 + w + \epsilon_1)\mathcal{H}\dot{\Phi} - (w + \epsilon_2)\nabla^2\Phi + \epsilon_3\mathcal{H}^2\Phi = 0 \quad (7)$$

with quantum corrections $\epsilon_i$. The equation is sensitive to the gravitational part of perturbation equations whose correction terms, derived here for the first time, require quantum gravity. This equation has a characteristic implication: classically, $\epsilon_i = 0$ and the last term cancels exactly, but with quantum corrections $\epsilon_3 = -2\alpha''\bar{p}^2/\alpha$ is negative. These corrections are usually small compared to the term $\nabla^2\Phi$, but can become important for modes of small comoving wave number $k$ such that $ak \ll H$. Such modes are outside the Hubble radius for which classically $\Phi$ would be preserved. With quantum corrections, however, such curvature perturbations are no longer conserved. This is seen by solving the equation $\ddot{\Phi} + (1 + \nu)\dot{\Phi}/\eta + \epsilon_3\Phi/\eta^2 = 0$, with $\nu = (5 + 3w)/(1 + 3w)$, for the behavior of large scale modes in conformal time $\eta$, approximating $w$ and $\epsilon_3$ by a constant for the qualitative effect: $\Phi(\eta) = \eta^\lambda$ with $\lambda = -\frac{\nu}{2} \pm \frac{1}{2}\sqrt{\nu^2 - 4\epsilon_3}$. Classically, there is one decaying mode and a constant one, corresponding to conserved $\Phi$. With nonzero $\epsilon_3$, however, the constant mode disappears, affecting the power spectrum. Since not only the magnitude and possibly the sign of $\epsilon_3$ but also the cosmic evolution time depend on the mode, a specific running of the spectral index can be expected. We comment here only on the magnitude of corrections for large scale modes which were created early in inflation: as before, an estimate for $\alpha - 1$ gives $1 \gg |\epsilon_3| \gg 10^{-6}$. During inflation, conformal time $\eta \propto e^{-H\tau}$ for modes currently visible on the largest scales changes by a factor $e^{-60}$, such that the constant classical solution is corrected by a factor $e^{-60\epsilon_3} \approx 1\text{–}10^2\epsilon_3$. On the lower end of $\epsilon_3$ this would not be observable soon, but with lattice areas expected to be much closer to the Planck scale, $\epsilon_3$ should be closer to one and the magnification due to the number of $e$-foldings further enhances quantum corrections to become potentially observable.

We have provided first effective cosmological perturbation equations which include correction terms from quantum gravity. The scheme of the derivation is systematic and general enough to include also other modes and backgrounds. Unlike low energy effective theory which is usually used to introduce quantum corrections in classical equations, effective theory taking into account the spatial discreteness expected from quantum gravity has revealed new effects whose magnitude can be much larger than expected on dimensional grounds. In addition, effects can be enlarged due to long cosmic evolution. A precise determination of the magnitude of corrections requires more detailed solutions of lattice states. This is accessible, possibly aided by numerical schemes, in the framework now provided by loop quantum gravity.

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