# Superconductor-Metal Transition in an Ultrasmall Josephson Junction Biased by a Noisy Voltage Source

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(Received 24 July 2006; published 16 January 2007)

Shot noise in a voltage source changes the character of the quantum (dissipative) phase transition in an ultrasmall Josephson junction: The superconductor-insulator transition transforms into the superconductor-metal transition. In the metallic phase, the *IV* curve probes the voltage distribution generated by shot noise, whereas in the superconducting phase, it probes the counting statistics of electrons traversing the noise junction.

DOI: 10.1103/PhysRevLett.98.030601 PACS numbers: 74.78.Na, 05.40.Ca, 74.50.+r

The phenomenon of noise is continuing to be in the focus of modern mesoscopic physics. A lot of attention is devoted to shot noise [1], in particular, its full counting statistics, non-Gaussian character, and asymmetry (odd moments) [2–4]. They are also objects of intensive experimental investigations [5,6]. This stimulated development of effective methods of noise detection [7–16]. It has been shown theoretically and experimentally that Coulomb blockade of a Josephson junction is very sensitive to noise from an independent source [8,10,11,15]. This can be used for noise spectroscopy. The source of shot noise was a current through an additional (noise) junction. It generated the voltage drop on the shunt, which was added to the voltage bias on the Josephson junction. Experimentally, this phenomenon was investigated at low noise currents, when electron tunneling events produced a sequence of voltage pulses on the shunt resistance well separated in time [8,10], whereas in the theory, only the case of low voltage bias was solved analytically [11]. The preliminary analysis of the case of high currents through the noise junction [15] has pointed out that the response of the Josephson junction to the voltage bias generated by this current must be essentially different from that to the ideal voltage bias. But quantitatively, the full response (the *IV* curve) was not analyzed, and the absence of the analysis of the noise effect on the Josephson junction in the low-impedance environment did not allow to derive conclusions about how noise can modify the quantum phase transition.

This Letter investigates *IV* curves of an ultrasmall Josephson junction biased both with the ideal and the noisy voltage source. The analysis has been done for the high- and low-impedance environment, characterized by the ratio $\rho = R/R_Q$. Here $R$ is the shunt resistance and $R_Q = h/4e^2 = \pi\hbar/2e^2$ is the quantum resistance for Cooper pairs. The most important outcome of the analysis is that noise modifies the character of the quantum (dissipative) phase transition: The well known "superconductor-insulator" transition [17–19] at $\rho = 1$ transforms to the "superconductor-metal" transition. This means that in the high-impedance phase $\rho > 1$, the zero-bias conductance does not vanish, but remains finite as in a junction between two normal metals.

Figure 1 shows the electric circuit discussed in the paper. Two voltages can bias the Josephson junction of capacitance $C_J$: (i) the constant voltage $V$ (ideal voltage bias), and (ii) the fluctuating voltage drop $V_s$ at the shunt resistance $R$. The average voltage $\bar{V}_s = I_s R$ is determined by the average current $I_s$ through the additional noise normal junction of capacitance $C_s$ and resistance $R_T$. It is assumed that $R_T \gg R$. Then $I_s = \tilde{V}/R_T$, the noise junction is voltage biased and tunneling events at the junction are governed by the Poissonian statistics.

If only the ideal voltage bias is used ($I_s = 0$), the $P(E)$ theory of incoherent tunneling of Cooper pairs yields the following current through the Josephson junction [17–19]:

$$I = \frac{\pi e E_J^2}{\hbar}[P_0(2eV) - P_0(-2eV)], \tag{1}$$

where the $P(E)$ function

$$P_0(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt \exp\left[J_0(t) + \frac{iEt}{\hbar}\right], \tag{2}$$

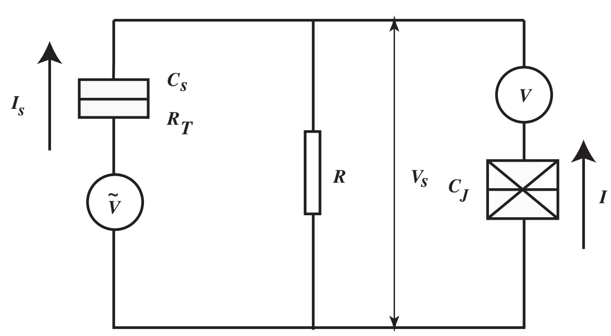


FIG. 1. Electric circuit with two sources of constant voltages $V$ and $\tilde{V}$. The Josephson junction with the capacitance $C_J$ is biased with the voltage $V$ and the fluctuating voltage drop $V_s$ at the shunt. The latter originates from the average current $I_s = \tilde{V}/R_T$ through the noise junction of capacitance $C_s$.

 

characterizes the probability to transfer the energy $E > 0$ to environment (or to absorb the energy $|E|$ from environment if $E < 0$). We restrict ourselves with the zero-temperature limit when the phase-phase correlator, which determines the $P(E)$ function, is given by [11,15]

$$\begin{aligned} J_0(t) &= \langle[\varphi_0(t) - \varphi_0(0)]\varphi_0(0)\rangle \\ &= \rho\bigg[-e^{t/\tau}\mathrm{E}_1\Big(\frac{t}{\tau}\Big) - e^{-t/\tau}\mathrm{E}_1\Big(-\frac{t}{\tau} + i0\Big) \\ &\quad - 2\ln\frac{t}{\tau} - 2\gamma - i\pi\bigg], \end{aligned} \tag{3}$$

where $\tau = RC$ is the relaxation time in the electric circuit, $C = C_J + C_s$, $\gamma = 0.577$ is the Euler constant, and $\mathrm{E}_1(z) = \int_1^\infty e^{-zt}dt/t$ is the exponential integral. The $P(E)$ theory is based on the time-dependent perturbation theory with respect to the small Josephson coupling energy $E_J$. Therefore, the current $I$ is small, and any feedback of the Josephson junction on the circuit and the noise is ignored. In addition, it is assumed that phase fluctuations are Gaussian. At $T = 0$, $P(E)$ vanishes for $E < 0$ since it is the probability of the transfer of the energy $|E|$ from the environment to the junction, which is impossible if $T = 0$. The subscript 0 points out that the phase fluctuations $\varphi_0$ and the $P_0(E)$ function are determined by the equilibrium Johnson-Nyquist noise.

If the ideal voltage bias $V$ is supplemented with the fluctuating voltage drop $V_s$ at the shunt resistance $R$, the $P_0(E)$ function in the expression for the current, Eq. (1), should be replaced by a more general function [11,15], which depends on the ideal bias $V$ and the averaged fluctuating bias $\bar{V}_s$:

$$P(2eV, 2e\bar{V}_s) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt e^{J_0(t)} e^{i2eVt/\hbar}\langle\exp[i\Delta\varphi_s(t)]\rangle. \tag{4}$$

The phase difference $\Delta\varphi_s(t) = \varphi_s(t) - \varphi_s(0) = (2e/\hbar)\times\int_0^t V_s(t)$ is determined by the fluctuating voltage $V_s(t)$. The $P(E)$ function can be presented as a convolution of the two $P(E)$ functions [20]:

$$P(2eV, 2e\bar{V}_s) = 2e\int_{-\infty}^{\infty} dV_s P_0(2e[V - V_s])P_s(2eV_s), \tag{5}$$

where $P_0(E)$ is related to equilibrium noise [Eq. (2)], and

$$P_s(E) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} dt e^{iEt/\hbar}\langle\exp[i\Delta\varphi_s(t)]\rangle, \tag{6}$$

is the $P(E)$ function for shot noise.

The averaged phase correlator in the expression for the shot-noise $P_s(E)$ function, Eq. (6), is a value of the generating function $\langle\exp[\xi\Delta\varphi_s(t)]\rangle$ at $\xi = i$. The generating function determines all moments and cumulants of the random phase difference $\Delta\varphi_s(t)$. For an ideal voltage bias of the noise junction, the generating function can be determined exactly [15,21] keeping in mind that the electrons traversing the noise junction produce a sequence of random voltage pulses at the shunt resistance $R$:

$$V_s(t) = \mathrm{sgn}(I_s)(e/C)\sum_i \Theta(t - t_i)e^{-(t-t_i)/\tau}, \tag{7}$$

where $t_i$ are random moments of time when an electron crosses the junction. The average time interval between tunneling events is $e/I_s$, and the number of the events in a fixed time interval is governed by Poissonian statistics. The voltage pulses generate the sequence of phase jumps:

$$\varphi_s(t) = \mathrm{sgn}(I_s)\pi\rho\sum_i \Theta(t - t_i)[1 - e^{-(t-t_i)/\tau}]. \tag{8}$$

The generating function for the random phase difference $\Delta\varphi_s(t)$ is given by

$$\langle\exp[\xi\Delta\varphi_s(t)]\rangle = \exp\bigg[\frac{I_s\tau}{e}\Phi(\xi, t)\bigg], \tag{9}$$

where

$$\begin{aligned} \Phi(\xi, t) = e^{\pi\rho\xi}&[\mathrm{E}_1(\pi\rho\xi e^{-t/\tau}) - \mathrm{E}_1(\pi\rho\xi)] - \frac{t}{\tau} - \gamma \\ &- \ln[-\pi\rho\xi(1 - e^{-t/\tau})] \\ &- \mathrm{E}_1[-\pi\rho\xi(1 - e^{-t/\tau})]. \end{aligned} \tag{10}$$

In the high-impedance case $\rho \gg 1$, the main contribution to the time integral in Eq. (4) comes from times $t \sim RC/\rho = R_Q C$ much shorter than $\tau = RC$ (see discussion in Ref. [15]). Then the voltage does not vary essentially during the time interval $t$, i.e., $\Delta\varphi_s(t) \approx 2eV_s t/\hbar$, and the expression for $\Phi(\xi, t)$ can be simplified:

$$\Phi(\xi, t) = -\mathrm{E}_1(-\pi\rho\xi t/\tau) - \gamma - \ln(-\pi\rho\xi t/\tau). \tag{11}$$

This means that the full statistics of the phase difference is identical to the full statistics of voltage fluctuations as found in Ref. [21]. Indeed, for the sequence of random voltage pulses giving by Eq. (7), the generating function for probability of voltage drop at the shunt is

$$F(\nu) = \langle\exp[\nu V_s C/e]\rangle = \exp\bigg[\frac{I_s\tau}{e}\Phi_v(\nu)\bigg], \tag{12}$$

where

$$\Phi_v(\nu) = -\mathrm{E}_1(-\nu) - \gamma - \ln(-\nu). \tag{13}$$

One can see that $\Phi(\xi, t)$ given by Eq. (11) is identical to $\Phi_v(\nu)$ in Eq. (13) with $\nu = \pi\rho\xi t/\tau$. Altogether, this means that the voltage distribution generated by shot noise and parameterized by the averaged bias $\bar{V}_s$,

$$p(V_s, \bar{V}_s) = \frac{C}{2\pi e}\int_{-\infty}^{\infty} dx e^{-ixV_sC/e}F(ix), \tag{14}$$

directly determines the shot-noise $P(E)$ function:

$$P_s(E) = \frac{1}{2e}p(-E/2e, \bar{V}_s). \tag{15}$$

Thus, the total $P(E)$ function is the $P(E)$ function for the equilibrium noise averaged over the voltage distribution generated by the noise current:

$$P(2eV, 2e\bar{V}_s) = \int_{-\infty}^{\infty} dV_s P_0(2e[V + V_s]) p(V_s, \bar{V}_s). \tag{16}$$

This approach called "the time-dependent $P(E)$ theory" has already been used in previous numerical simulations [22]. The approach is justified for the high-impedance environment $\rho \gg 1$, but is not valid in the opposite case of the low-impedance environment $\rho < 1$ (see below).

The high-impedance limit $\rho = R/R_Q \to \infty$ of the equilibrium $P(E)$ function is the $\delta$-function: $P_0(2eV) = \delta(2eV - 2e^2/C)$ [23]. Then Eq. (16) yields

$$P(2eV, 2e\bar{V}_s) = \frac{1}{2e} p\left(\frac{e}{C} - V, \bar{V}_s\right). \tag{17}$$

and the $P(E)$ function (current) directly scans the voltage probability distribution generated by shot noise and given by Eq. (14). The dependence of this function on $\bar{V}_s$ at $V = 0$ is plotted in Fig. 2. In contrast to the ideal voltage bias, which in the limit $\rho \to \infty$ yields the sharp peak $P(2eV) = (1/2e)\delta(V - e/C)$, the noisy voltage bias yields the non-Gaussian broad maximum. At $V = 0$, the $P(E)$-plot scans dependence of the voltage distribution $p(V_s, \bar{V}_s)$ on $\bar{V}_s$ at fixed $V_s = e/C$. In order to scan dependence on $V_s$, one should tune the ideal bias $V$.

Let us consider now the $P(E)$-dependence on $V$ at weak noisy bias $\bar{V}_s \ll e/C$. Linearizing Eq. (14) with respect to $\bar{V}_s$ and taking the integral by parts one obtains

$$\begin{aligned} p(V_s, \bar{V}_s) &= \frac{\bar{V}_s C^2}{2\pi e^2} \int_{-i\infty}^{i\infty} d\nu e^{-\nu V_s C/e} \Phi_v(\nu) \\ &= \frac{\bar{V}_s C}{2\pi e V_s} \int_{-\infty}^{\infty} e^{-ixV_s C/e} \frac{e^{ix} - 1}{ix} dx \\ &= \frac{\bar{V}_s C}{2\pi e V_s} \\ &\quad \times \int_{-\infty}^{\infty} \frac{\sin[x(1 - V_s C/e)] + \sin(x V_s C/e)}{x} dx. \end{aligned} \tag{18}$$

This yields $p(V_s, \bar{V}_s) = \bar{V}_s C/eV_s$ at $0 < V_s < e/C$ and $p(V_s, \bar{V}_s) = \bar{V}_s C/2eV_s$ if $V_s = e/C$ exactly. At $V_s > e/C$, $p(V_s, \bar{V}_s)$ vanishes. This is the voltage distribution $p(V_s) \propto dt/dV_s$ inside a single voltage pulse $V_s(t) = (e/C)\exp(-t/\tau)$. Inserting this voltage distribution into Eq. (17) yields the $P(E)$ function $[= P(2eV, 2e\bar{V}_s]$ plotted as a function of $V$ in Fig. 3. The singularity $1/V_s$ in $p(V_s, \bar{V}_s)$ leads to the singularity $1/(V - e/C)$ in $P(E)$. The initial value of $P(E)$ at $V = 0$ (due to finite bias $\bar{V}_s$) is proportional to the ratchet current $I_0$ found analytically in Ref. [11] [see Eq. (21) there]. As for the vertical jump of $P(E)$, it is a consequence of the approximation of $P_0(E)$ by the delta-function leading from Eq. (16) and (17). However, even for large $\rho$, the peak of $P_0(E)$ has a finite width of the order of $e/C\sqrt{\rho}$ (ignoring a logarithm factor) [19]. This smears the jump and leads to a finite linear slope at $V = 0$ (the dashed line in Fig. 3) at $V = 0$, as evident from Eq. (16). This slope corresponds to metallic conductance $dI/dV$ analytically calculated in Ref. [11] for $\rho \gg 1$ [see Eq. (20) there]. Note that this conductance differs from the conductance $dI/d\bar{V}_s = I_0/\bar{V}_s$ proportional to the linear slope at $\bar{V}_s = 0$ in Fig. 2, but both are signatures of the metallic behavior.

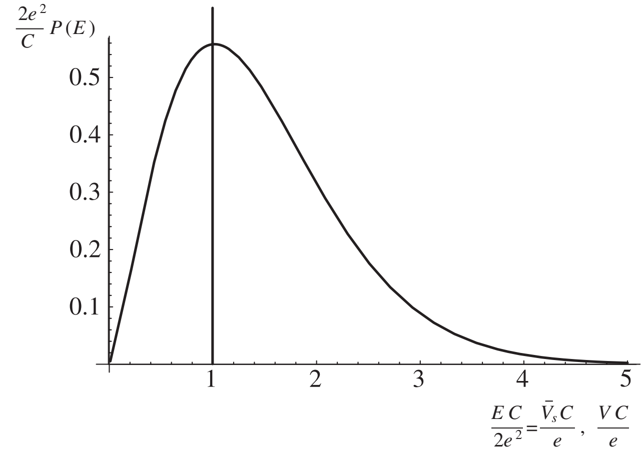


FIG. 2. $P(E)$ function in the high-impedance limit $R \gg R_Q$ as a function of the noisy bias $\bar{V}_s$ $[P(0, 2eV_s)]$, or of the ideal bias $V$ $[P(2eV, 0) = (1/2e)\delta(V - e/C)]$.

In the opposite limit of low-impedance environment $\rho \ll 1$, the most important contribution to the $P(E)$ function comes from long times $t \gg \tau$, and the $P(E)$ function does not scan the voltage distribution anymore. In this limit, the Johnson-Nyquist correlator, Eq. (3), becomes

$$J_0(t) = -2\rho\left(\ln\frac{t}{\tau} + \frac{i\pi}{2}\right), \tag{19}$$

and the logarithm of the generating function, Eq. (10), is approximately given by

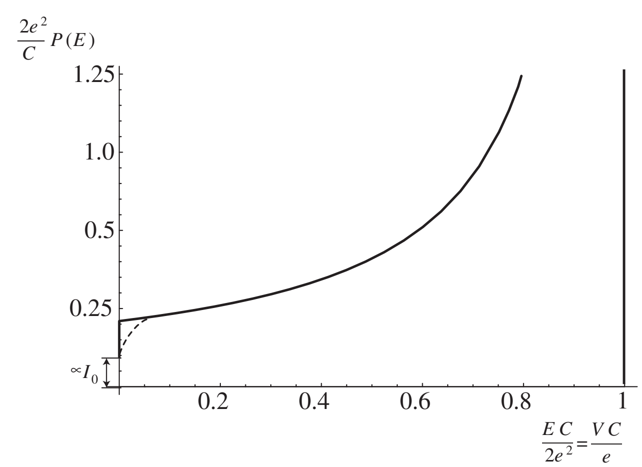


FIG. 3. $P(E)$ function at weak noise current ($\bar{V}_s C/e = 0.15$). The dashed line shows smearing of the zero-bias jump with finite width of the peak in the equilibrium $P_0(E)$ function.

$$\Phi(\xi, t) = \frac{t}{\tau}(e^{\pi\rho\xi} - 1). \tag{20}$$

This expression corresponds to the Poissonian statistics of phase jumps, identical to the Poissonian full counting statistics [15,21]. Since $\rho \ll 1$, we can expand in $\rho$. At the same time, we can generalize this expansion on other possible types of statistics. Then the phase correlator (the generating function at $\xi = i$) can be written as

$$\langle \exp[i\Delta\varphi_s(t)]\rangle = \exp\left[\frac{i2e\bar{V}_s t}{\hbar}\lambda\right], \tag{21}$$

where

$$\lambda = \lambda_R + i\lambda_I \approx 1 + \frac{i\pi\rho\xi}{2}\frac{\langle\langle n^2\rangle\rangle}{\langle\langle n\rangle\rangle} - \frac{\pi^2\rho^2\xi^2}{6}\frac{\langle\langle n^3\rangle\rangle}{\langle\langle n\rangle\rangle}. \tag{22}$$

Here $\langle\langle n^k\rangle\rangle$ are cumulants of the full counting statistics, $n$ being a number of electrons traversing the noise junction during the time $t$. For the Poissonian statistics, all cumulants are equal to the first cumulant $\langle\langle n\rangle\rangle$, which is the average number of electrons traversing the junction. Inserting the expression (21) into Eq. (4), one obtains the $P(E)$ function in the low-impedance limit:

$$P(2eV, 2e\bar{V}_s) = \frac{2}{\pi\hbar}\int_0^\infty dt\left(\frac{\tau}{t}\right)^{2\rho}\sin(\pi\rho) \times \mathrm{Im}\left\{\exp\left[i\left(\frac{2eVt}{\hbar} + \frac{2e\bar{V}_s t}{\hbar}\lambda\right)\right]\right\} \approx \frac{2\tau\rho}{\hbar}\left[\frac{\hbar}{2e\tau(V + \bar{V}_s|\lambda|)}\right]^{1-2\rho}. \tag{23}$$

Though one cannot use this expression at very low voltages where the perturbation theory in $E_J$ becomes invalid [24], the "superconductivity" current peak is present anyway, both for ideal and noisy voltage bias. Equation (23) demonstrates that in low-impedance environment, the $IV$ curve probes the counting statistics even though the dependence on high cumulants $k > 2$ is not so pronounced.

Though the present Letter is restricted with $T = 0$, on the basis of it, one may suggest that the metallic state would be less sensitive to the temperature than the insulator state. Indeed, one can probe the insulator behavior in the limit $V \to 0$ only under conditions $T \ll V \ll e/C$. Meanwhile, for detection of metallic conductance, a weaker restriction $T \ll e/C$ on the temperature is sufficient.

In summary, shot noise in the voltage source dramatically changes the character of the quantum (dissipative) phase transition in the ultrasmall Josephson junction tuned by the environment impedance. For the ideal voltage bias, this is a transition from the superconducting state ($\rho = R/R_Q < 1$) to the insulator (Coulomb blockade, $\rho = R/R_Q > 1$). In contrast, in the case of the noisy voltage source, the transition takes place between the superconducting phase and the metallic phase with finite zero-bias conductance. This transition can be called superconductor-metal transition. In the metallic phase, the $IV$ curve is a probe of the voltage distribution generated by shot noise, whereas in the superconducting phase, the $IV$ curve is probing the counting statistics for electrons traversing the noise junction.

The author thanks Pertti Hakonen and Yuli Nazarov for interesting discussions.

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