

Eigenfunction Fractality and Pseudogap State near the Superconductor-Insulator Transition

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(Received 9 October 2006; published 8 January 2007)

We develop a theory of a pseudogap state appearing near the superconductor-insulator (SI) transition in strongly disordered metals with an attractive interaction. We show that such an interaction combined with the fractal nature of the single-particle wave functions near the mobility edge leads to an anomalously large single-particle gap in the superconducting state near SI transition that persists and even increases in the insulating state long after the superconductivity is destroyed. We give analytic expressions for the value of the pseudogap in terms of the inverse participation ratio of the corresponding localization problem.

DOI: 10.1103/PhysRevLett.98.027001

PACS numbers: 74.78.-w

A rapidly growing number of experiments [1–9] on various disordered superconductors shows that a novel phase often appears on the insulating side of the superconductor-insulator transition. As the disorder strength is increased the superconductivity is suppressed leading to a strange insulator characterized by a large thermally assisted resistance with a small but hard gap (Fig. 1). Experimentally, the phase diagram of disordered superconductors is often explored by varying the applied magnetic field. On the superconducting side of the transition a relatively small field destroys the superconductivity resulting in a hard-gap insulating state. At larger magnetic fields the resistance and the gap drop [4–8]. This is observed only in a narrow window of disorder strengths, away from this window on a superconducting side the application of magnetic field converts the superconductor into a normal metal as usual.

It is tempting to explain these data by the formation of localized Cooper pairs [4,10]. In this picture the superconductivity is due to a fragile coherence between localized Cooper pairs, while the energy to break the pair is much larger and remains finite even when the coherence (and thus the superconductivity) is destroyed. The hypothesis of preformed Cooper pairs is further confirmed by the behavior of these superconductors at higher temperatures. On the insulating side of the transition in thick (effectively 3D) films one observes [1,3] Arrhenius temperature behavior of the resistivity, $R(T) \propto \exp(T_I/T)$, at low temperatures. The experimental value of the activation energy T_I is somewhat larger than the superconducting gap in less disordered samples and grows with the disorder [1,3]. However, at higher temperatures this behavior is replaced [3] by Mott's variable-range hopping $R(T) \sim \exp(T_M/T)^{1/4}$. This can be understood if the insulating pseudogap is due to preformed Cooper pairs with a relatively large pairing energy T_I . In this Letter we show that preformed Cooper pairs appear in the semimicroscopic three-dimensional model that contains only low energy

electrons with weak BCS-type attraction and a strong random potential that leads to Anderson localization of single-particle states.

In the presence of preformed Cooper pairs the parity effect should arise—the ground-state energy with even number of electrons is lower than that for the closest odd number. The corresponding *parity gap* $\Delta_p \sim T_I \sim \delta / \ln(\delta/\Delta)$ has been calculated by Matveev and Larkin [11] for small superconducting grains. Here $\delta = 1/(\nu_0 L^3)$ is the mean level spacing in the grain, $\Delta \ll \delta$ is the energy gap in the bulk superconductor, and ν_0 and L^3 are the density of orbital states and the volume of the grain, respectively. In this Letter we argue that the result of Ref. [11] can be generalized to bulk Anderson insulators. In this case, L is replaced by the localization radius L_{loc} , so that $\delta \rightarrow \delta_L = 1/(\nu_0 L_{loc}^3)$, and the BCS Δ is replaced by the superconducting gap Δ_{crit} at the Anderson transition point. The fractal nature of near-critical wave functions characterized by the fractal dimension $D_2 < 3$ determines the reduction factor $(\delta_L/\Delta_{crit})^{1-D_2/3} \gg 1$ that replaces $\ln(\delta/\Delta)$ in the Matveev-Larkin formula.

We consider two different regimes. In more disordered materials where $\delta_L \gg \Delta_{crit}$ the Cooper instability and

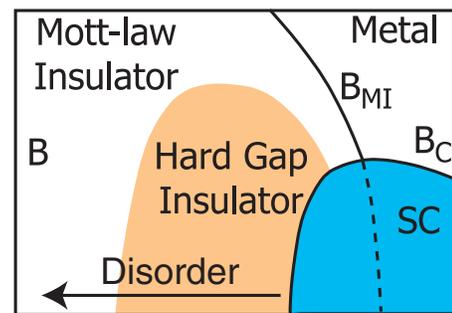


FIG. 1 (color online). Schematics of low-temperature phase diagram of disordered superconductors in the vicinity of the superconductor-insulator transition.

superconductive long-range order disappear. However, the attraction between electrons persists as long as δ_L remains smaller than the Debye frequency ω_D and results in the “local” pairing of electrons with opposite spins occupying the same localized state. We show below that in this regime the *hard-gap insulator* is formed with properties similar to those observed in [1,3]. We further argue that in the less localized regime, when $\delta_L \leq \Delta_{\text{crit}}$, the unusual superconductive state with a pseudogap is formed. The features of this state are (i) single-electron excitation gap Δ_1 is larger than superconductive gap Δ so that the ratio Δ_1/T_c is anomalously high, and (ii) insulating trend in the $R(T)$ curves exists above T_c .

We assume that superconductivity is due to attraction between electrons that originates at high energy scales $\sim \omega_D$ that is not affected by localization of electron wave functions for moderate disorder $\delta_L \leq \omega_D$. In a fermion system with weak attraction one can leave only pair interaction terms in the Hamiltonian leading to the usual BCS model in the basis of localized electron states [12]:

$$H = \sum_{j\sigma} \epsilon_j c_{j\sigma}^\dagger c_{j\sigma} - \frac{\lambda}{\nu_0} \sum_{j,k} M_{jk} c_{j\uparrow}^\dagger c_{j\downarrow}^\dagger c_{k\downarrow} c_{k\uparrow}, \quad \text{where} \quad (1)$$

$$M_{jk} = \int d\mathbf{r} \psi_j^2(\mathbf{r}) \psi_k^2(\mathbf{r}),$$

λ is dimensionless Cooper coupling constant, ϵ_j is the single-particle energy of the state j , and $c_{j\sigma}$ is the corresponding electron operator for the spin projection σ .

Physical properties of the electron system are controlled by the electrons near the Fermi level, so a very important implicit ingredient of the model (2) is the statistics of matrix elements M_{jk} between eigenstates in the vicinity of Anderson mobility edge. The key feature of these nearly critical wave functions is their fractal structure [13] that shows in the anomalous scaling of diagonal matrix elements $M_{jj} \equiv M_j$ (“inverse participation ratios,” IPRs) with localization length: typical IPR $\bar{M} \propto L_{\text{loc}}^{-D_2}$, where fractal dimension $D_2 < 3$. Numerical studies [14,15] indicate that $D_2 = 1.30 \pm 0.05$ for the standard 3D Anderson transition. The IPR distribution function $\mathcal{P}(M_j)$ has been studied in [15]. Scaling theory of localization predicts that near the mobility edge $\mathcal{P}(M_j)$ acquires a scale-invariant form and this is indeed what was observed [15]. The same data demonstrate that $\mathcal{P}(M_j)$ decreases fast for atypically extended states; i.e., at $M_j/\bar{M} \ll 1$. This allows us to use the typical value

$$\bar{M} = L_0^{-3} (L_{\text{loc}}/L_0)^{-D_2}, \quad (2)$$

where L_0 is the short-scale cutoff length of the fractal behavior. The associated energy scale $E_0 = 1/(\nu_0 L_0^3)$ depends on the microscopic details of the model of disorder and might be small compared to Fermi-energy E_F . Localization length depends on Fermi-energy (in the scaling region $L_{\text{loc}} \gg L_0$) as $L_{\text{loc}} \approx L_0 [E_0/(E_m - E_F)]^\nu$,

where E_m is the position of the mobility edge and ν is the localization length exponent.

Another important property of the nearly critical eigenstates is their strong correlation in energy and real space [13,16,17] even in the limit of strong fractality $D_2 \ll 3$. It results in the scaling dependence of the average matrix elements \bar{M}_{jk} on the energy difference $E_j - E_k = \omega$:

$$\mathcal{V} \bar{M}_{jk} \equiv M(\omega) \approx \begin{cases} (L_{\text{loc}}/L_0)^{3\gamma} & \text{at } \omega \ll \delta_L \quad (\text{a}) \\ (E_0/\omega)^\gamma & \text{at } \delta_L \ll \omega \leq E_0 \quad (\text{b}) \end{cases} \quad (3)$$

where $\gamma = 1 - D_2/3$ and \mathcal{V} is the total system’s volume. Note that in the critical region $M(\omega) \gg 1$ in contrast both to a metal and to a deep insulator ($L_{\text{loc}} \sim L_0$), where $M(\omega) \approx 1$.

We begin with the insulating region $\Delta \ll \delta_L \ll \omega_D$ where Cooper interaction can be treated perturbatively. In the first order of the perturbation theory we take into account only diagonal terms $j = k$ of the interaction similar to the case of ultrasmall grain [11]. Then the energy (counted from E_F) required to break a bound pair of electrons siting in the j th orbital state is $2\Delta_P^{(j)} = \frac{\lambda}{\nu_0} M_j$. Typical value of this “parity gap” (cf. [11]) scales then as

$$\Delta_P = \frac{\lambda}{2} E_0 \left(\frac{L_0}{L_{\text{loc}}} \right)^{D_2} \propto (E_m - E_F)^{\nu D_2} \quad (4)$$

Neglecting fluctuations in the local values of $\Delta_P^{(j)}$, one finds that all states occupied by single electrons are shifted up by the amount Δ_P that leads to the electron DOS $\tilde{\nu}(\varepsilon) = \nu_0 \theta(\varepsilon - \Delta_P)$. In fact, values of $\Delta_P^{(j)}$ differ for different localized states, and the average density of states $\nu(\varepsilon)$ in a large sample is determined by the IPR distribution $\mathcal{P}(M)$:

$$\nu(\varepsilon) = \nu_0 \int_0^{2\varepsilon\nu_0/\lambda} \mathcal{P}(M) dM. \quad (5)$$

As mentioned above, numerical data for $\mathcal{P}(M)$ indicate its very fast decrease at $M/\bar{M} \rightarrow 0$. Thus the DOS shape (5) is not far from a rectangular sharp gap, with the gap value given by Eq. (4). We emphasize that (i) parity gap Δ_P is much larger than level spacing δ_L at $L_{\text{loc}}/L_0 > (2/\lambda)^{1/(3-D_2)}$, and (ii) the DOS (5) does not contain any “coherence peak” above the gap (cf. Ref. [18]).

We associate the spectral gap Δ_P with the measured [1,3] activation energy T_I , assuming that hard-gap conductivity behavior is due to single-electron hopping (at the lowest temperatures variable-range hopping of localized pairs is expected to prevail). The external parameter $(E_m - E_F)$ representing the disorder strength in Eq. (4) can be replaced with an experimentally more accessible parameter $(\sigma_c - \sigma) \propto (E_m - E_F)$. Here σ is the high temperature conductivity and σ_c is the value of the conductivity where the parity gap Δ_P first develops. We obtain

$$T_I = A(1 - \sigma/\sigma_c)^{\nu D_2}, \quad (6)$$

where A is conductivity-independent. This equation pre-

dicts a moderate increase of T_I with disorder strength in agreement with the experimental data [1]; see Fig. 2.

We now turn to the parameter region $\delta_L \ll \Delta$ where one expects a global superconductive coherence to exist at low enough T . Indeed, in this regime a given localized single-particle state typically overlaps in a real space with a large number $\sim \Delta/\delta_L$ of eigenstates ψ_j in the same energy strip $|\epsilon_j| \leq \Delta$. It is natural to expect that in this case the mean-field approximation (MFA) is qualitatively correct. To test the validity of MFA, we compared its prediction for $T_c(\delta_L)$ (see below) with the transition temperature that was found numerically by computing the first terms of the virial expansion applied to (1) with M_{ij} determined by exact diagonalization. Reasonably good agreement was found [19]. To proceed with the MFA, we introduce averaged energy-dependent pairing amplitudes $F(\epsilon_j) = \langle c_{j\uparrow}c_{j\downarrow} \rangle$ and the gap function $\Delta(\epsilon) = \lambda \int d\epsilon_1 M(\epsilon - \epsilon_1) F(\epsilon_1)$. Following standard steps, we decouple interaction term in the Hamiltonian (2) via the gap function $\Delta(\epsilon)$, calculate anomalous averages $F(\epsilon)$, and arrive at the modified BCS gap equation in the form

$$\Delta(\epsilon) = \frac{\lambda}{2} \int_{-\infty}^{+\infty} d\epsilon_1 \frac{M(\epsilon - \epsilon_1)\Delta(\epsilon_1)}{\sqrt{\epsilon_1^2 + \Delta^2(\epsilon_1)}} \tanh \frac{\sqrt{\epsilon_1^2 + \Delta^2(\epsilon_1)}}{2T}. \quad (7)$$

Gap function $\Delta(\epsilon)$ obeying Eq. (7) is an even function of ϵ with the maximum value $\Delta(\epsilon = 0) \equiv \Delta_0$.

Superconducting transition temperature T_c is determined by linearization of Eq. (7) with respect to $\Delta(\epsilon)$. Because of power-law decrease of $M(\omega)$ at large arguments, the integral in Eq. (7) converges and is dominated by $\omega \sim T$ (“infrared superconductivity”) so no upper cut-off is needed, contrary to usual BCS problem. When the Fermi level is very close to the Anderson mobility edge E_m

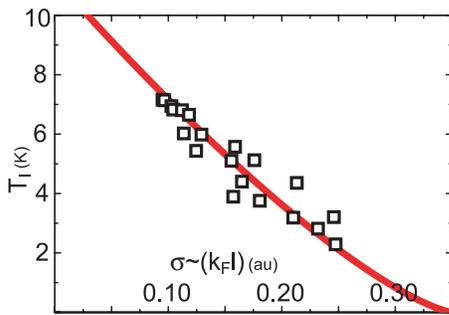


FIG. 2 (color online). Experimental values of the gap from Ref. [1], T_I (boxes) and a fit to the Eq. (6) with $\nu = 1$, $D_2 = 1.3$. The only fitting parameter was the constant $A = (\lambda/2)E_0$; the data points of Ref. [1] correspond to $E_0 \approx 100$ K at $\lambda \approx 0.2$ extracted from the BCS value of $T_c \approx 3$ K for less disordered samples [1] and $\omega_D \approx 500$ K. The value of σ_c was determined from high T data. Application of scaling formulas is justified by the large value of the localization length: $L_{\text{loc}}^{\text{min}} > 30$ Å which was deduced from the Mott temperature characterizing the resistivity of similar samples at intermediate temperatures [3].

and level spacing $\delta_L \sim E_0[(E_m - E_F)/E_0]^{3\nu}$ is negligibly small, one can use for $M(\omega)$ line (b) of Eq. (3). Then the critical temperature is given by

$$T_c^0(\lambda, \gamma) = E_0 \lambda^{1/\gamma} C(\gamma), \quad (8)$$

where dimensionless function $C(\gamma)$ can be computed numerically. At small λ the value given by Eq. (8) exceeds the BCS value $T_{\text{BSC}} \sim \omega_D e^{-1/\lambda}$. This may lead to a maximum in T_c near the critical disorder. The zero-temperature energy gap $\Delta_{T=0}$ in the same limit $\delta_L \rightarrow 0$ is given by Eq. (8) with $C(\gamma)$ replaced by another function $D(\gamma)$. We plot $D(\gamma)$ and $2D(\gamma)/C(\gamma)$ in Fig. 3.

Using Eq. (8) we eliminate the interaction constant λ and the cutoff parameter E_0 from Eq. (4) and arrive at

$$\Delta_P = \frac{1}{2C\gamma} \delta_L (T_c^0/\delta_L)^\gamma \quad (9)$$

This formula (applied at $\delta_L \gg T_c^0$) generalizes the results of Ref. [11] to bulk strongly disordered superconductors. In contrast to Ref. [11] here the reduction of Δ_P compared to δ_L is not due to the renormalization of attractive interaction (which is absent) but to the enhancement of the matrix elements M_j due to fractality.

To study the effect of nonzero level spacing δ_L upon T_c , we approximate $M(\omega)$ by a simple interpolation formula $M(\omega) = E_0^\gamma (\omega^2 + \delta_L^2)^{-\gamma/2}$ and solve the linearized version of Eq. (7) for $T_c(\delta_L)$ numerically. Since $M(\omega)$ is a uniform function of T and δ_L , while the coupling constant λ and E_0 enter Eq. (7) only in a combination $E_0 \lambda^{1/\gamma}$, it is possible to present the dependence $T_c(\delta_L)$ in the form

$$T_c(\delta_L) = T_c^0 \mathcal{T}_\gamma \left(\frac{T_c^0}{\delta_L} \right) \quad (10)$$

where T_c^0 is defined in Eq. (8) and scaling function $\mathcal{T}_\gamma(x)$ does not depend on λ . This universal function was found numerically for $\gamma = 0.57$ (corresponding to $D_2 = 1.3$ for 3D Anderson transition), the result is plotted in Fig. 3. The actual T_c is suppressed as compared to the mean-field result due to fluctuations.

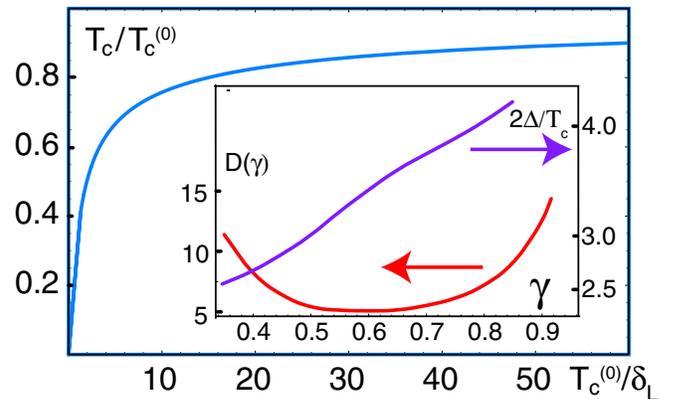


FIG. 3 (color online). Suppression of T_c as a function of level spacing within localization volume. The inset shows γ dependence of the dimensionless gap and $2\Delta(0)/T_c$ for $\delta_L = 0$.

Although gap Eq. (7) is similar to the conventional one, the real-space properties of the state that it describes are unusual. The local pairing amplitude $F(\mathbf{r}) = \sum_j \langle c_{j\uparrow} c_{j\downarrow} \rangle \psi_j^2(\mathbf{r})$ is extremely inhomogeneous in space, populating only small fraction $\propto [T_c(\delta_L)/E_0]^\gamma$ of the total volume [19]. Diamagnetic response of such a superconductor differs strongly from that of usual “dirty-limit” materials with uniform $|F(\mathbf{r})|$, but reminds that of weakly coupled Josephson junctions arrays. Qualitatively, we expect (i) extremely weak Meissner effect and considerably stronger linear shielding effect, and (ii) superconductor \rightarrow gaped insulator \rightarrow Mott insulator sequence of transitions upon magnetic field increase.

Next, we discuss the effect of “local pairing” considered previously [see Eq. (4)] in insulating state. We have seen that in insulating state when $\delta_L \gg \Delta$ single-particle excitation (carrying spin $\frac{1}{2}$) have a gap $\Delta_1 = \Delta_p$. Excitations that involve only hopping of paired electrons between localized levels and do not involve breaking pairs are gapless; i.e., the energy gap Δ vanishes. In superconducting state within MFA we have $\Delta_1 = \Delta + \Delta_p$, while the gap for pair excitations [20] (without pair breaking) is $\Delta_2 = 2\Delta$. When $L_{\text{loc}}/L_0 \rightarrow \infty$, the parity gap Δ_p becomes much larger than δ_L . Therefore, we expect that there is a regime $\Delta_p \gtrsim \Delta$ in the superconducting state where the spin gap Δ_1 is larger than the energy gap.

Note that it is the spin gap Δ_1 that is measured as a spectral gap in a superconductive state, via tunneling conductance or optical conductivity experiments. We thus conclude that superconductors near S-I transition are expected to have anomalously large ratio of spectral gap to transition temperature. Suppression of single-particle density of states described by Eq. (5) can be observed via tunneling conductance measurements. An additional suppression of T_c in comparison with Δ_p is due to electron-electron interaction in the density channel, not included into the model (1).

An anomalously large ratio Δ_1/T_c leads to the insulating trend of the resistivity versus temperature behavior in the intermediate temperature range $T_c < T \leq \Delta_1$. This was observed in strongly disordered superconductors [8]; it is also known as the pseudogap phenomenon in underdoped cuprates [21] where it shows in resistivity, NMR or angle-resolved photoemission spectroscopy (ARPES) data. The quantitative similarity between $R(T, B)$ behavior in InO_x films and underdoped cuprates [22,23] allows one to speculate that the pseudogap in underdoped cuprates might also be related to pairing of electrons on localized states. The important difference of the cuprates is the d-wave symmetry of the pairing.

In conclusion, weak Anderson insulators with Cooper attraction are shown to possess hard insulating gap whose magnitude is determined by the IPR statistics near the mobility edge. Although this gap is due to electron pairing, it does not lead to a coherence peak. In the ground-state of this insulator all electrons are paired on individual local-

ized eigenfunctions. When the Fermi level gets closer to the mobility edge, superconductive correlations develop between localized pairs. Key features of the predicted superconductive ground-state are extreme inhomogeneity of superconductive correlations in real space, an unusually large (compared to T_c) single-particle excitation gap (spin gap), and pseudogaped regime at temperatures about T_c . All these unusual features are due to the fractal nature of localized eigenstates near the mobility edge.

We are grateful to B.L. Altshuler, T.I. Baturina, E. Cuevas, A.M. Finkelstein, V.F. Gantmakher, A.S. Ioselevich, I.V. Lerner, A. Millis, A.D. Mirlin, M. Mueller, Z. Ovadyahu, V.V. Ryazanov, D. Shahar, and M.A. Skvortsov for useful discussions. This research was supported by NSF Grants No. DMR 0210575 and No. ECS-0608842, by EPSRC Grant No. GR/R95432/01, by RFBR Grants No. 04-02-16348, No. 06-02-16533, and by the program “Quantum Macrophysics” of RAS. E. Y. was supported by the A. P. Sloan Research Foundation and NSF Grant No. DMR 0547769.

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