Dynamics of Spin-$\frac{1}{2}$ Quantum Plasmas

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The fully nonlinear governing equations for spin-$\frac{1}{2}$ quantum plasmas are presented. Starting from the Pauli equation, the relevant plasma equations are derived, and it is shown that nontrivial quantum spin couplings arise, enabling studies of the combined collective and spin dynamics. The linear response of the quantum plasma in an electron-ion system is obtained and analyzed. Applications of the theory to solid state and astrophysical systems as well as dusty plasmas are pointed out.

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There is currently a great deal of interest in investigating collective plasma modes [1–8] in quantum plasmas, as such plasmas could be of relevance in nanoscale electromechanical systems [9–11], in microplasmas and dense laser plasmas [12], and in laser interactions with atomic systems [13,14]. For example, Refs. [1,3–5] used quantum transport models in order to derive modified dispersion relations for Langmuir and ion-acoustic waves, while Shukla and Stenflo [15] investigated drift modes in nonuniform quantum magnetoplasmas. Moreover, it is known that cold quantum plasmas can support new dust modes [16,17]. In Ref. [8], it was shown that electron quantum plasmas could support highly stable dark solitons and vortices. Further examples of quantum plasmas and the range of validity of their descriptions has been discussed recently in Ref. [18]. The above studies of quantum plasmas have used models based on the Schrödinger description of the electron. It is expected that new and possible important effects could appear as further quantum effects are incorporated in models describing the quantum plasma particles. The coupling of spin to classical motion has attracted interest in the literature (see, e.g., [19–31]). Much work has been done concerning single particle spin effects in external field configurations, such as intense laser fields [22–27], and the possible experimental signatures thereof. However, there has also been interest in excitations of collective modes in spin systems, such as spin waves, in a wide scientific community. For example, in Refs. [19–21] hydrodynamical models including spin were presented, and further theory concerning spin, angular momentum, and the forces related to spin was discussed in Refs. [29,30]. Moreover, spin waves in spinor Bose condensates have recently been discussed in, e.g., Ref. [31]. The treatment of charged particles and plasmas using quantum theory has received attention in astrophysical settings, especially in strongly magnetized environments [32,33]. For example, the effects of quantum field theory on the linear response of an electron gas have been analyzed [34], results concerning the spin dependence of cyclotron decay on strong magnetic fields have been presented [35], and the propagation of waves in strongly magnetized plasmas has been considered [36].

In this Letter, we present for the first time the fully nonlinear governing equations for spin-$\frac{1}{2}$ quantum electron plasmas. Starting from the Pauli equation describing the nonrelativistic electron, we show that the electron-ion plasma equations are subject to spin-related terms. These terms give rise to a multitude of collective effects, of which some are investigated in detail. Applications of the governing equations are discussed, and it is shown that under certain circumstances the collective spin effects can dominate the plasma dynamics.

We will assume that the electron wave function can be written in the product form $\Psi = \Psi_{(1)}\Psi_{(2)}\ldots\Psi_{(N)}$, where N is the number of particle states. Thus, we will here neglect the effects of entanglement and focus on the collective properties of the quantum electron plasma. Then the nonrelativistic evolution of spin-$\frac{1}{2}$ particles, as described by the two-component spinor $\Psi_{(\alpha)}$, is given by (see, e.g., [37]) $i\hbar\partial\Psi_{(\alpha)} = \hat{H}\Psi_{(\alpha)}$

$$\hat{H} = \left[-\frac{\hbar^2}{2m_e}\left(\boldsymbol{\nabla} + \frac{ie}{\hbar c}\boldsymbol{A}\right)^2 + \mu_B \boldsymbol{B}\cdot\boldsymbol{\sigma} - e\phi\right], \quad (1)$$

where α numbers the particle states, m_e is the particle mass, $\boldsymbol{A}$ is the vector potential, e is the magnitude of the electron charge, $\mu_B = -e\hbar/2m_ec$ is the electron magnetic moment, ϕ is the electrostatic potential, and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli spin matrices.

By introducing the decomposition of the spinors according to $\Psi_{(\alpha)} = \sqrt{n_{(\alpha)}}\exp(iS_{(\alpha)}/\hbar)\varphi_{(\alpha)}$, we may derive a set of N coupled fluid equations [37] for the densities $n_{(\alpha)}$, the velocities $\boldsymbol{v}_{(\alpha)} = (1/m_e)(\boldsymbol{\nabla}S_{(\alpha)} - i\hbar\varphi^\dagger\boldsymbol{\nabla}\varphi) + (e/m_ec)\boldsymbol{A}$, and the spin vectors $\boldsymbol{s}_{(\alpha)} = (\hbar/2)\varphi^\dagger_{(\alpha)}\boldsymbol{\sigma}\varphi_{(\alpha)}$ (where $\varphi_{(\alpha)}$ is the 2-spinor carrying the spin-$\frac{1}{2}$ properties).

Next we define the total particle density for the species with charge q according to $n_e = \sum_{(\alpha)=1}^N p_\alpha n_{(\alpha)}$, where p_α is the probability related to the wave function $\Psi_{(\alpha)}$. Using the ensemble average $\langle f\rangle = \sum_\alpha p_\alpha (n_{(\alpha)}/n_e) f$ for any tensorial quantity f, we define the total electron fluid velocity for charges $\boldsymbol{V}_e = \langle \boldsymbol{v}_{(\alpha)}\rangle$ and the total electron spin density $\boldsymbol{S} = \langle \boldsymbol{s}_{(\alpha)}\rangle$. From these definitions, we can define the microscopic velocity in the electron fluid rest frame according to $\boldsymbol{w}_{(\alpha)} = \boldsymbol{v}_{(\alpha)} - \boldsymbol{V}_e$, satisfying $\langle \boldsymbol{w}_{(\alpha)}\rangle = 0$, and the microscopic spin density $\boldsymbol{\mathcal{S}}_{(\alpha)} = \boldsymbol{s}_{(\alpha)} - \boldsymbol{S}$, such that $\langle \boldsymbol{\mathcal{S}}_{(\alpha)}\rangle = 0$.

We then obtain the conservation equations

$$\partial_t n_e + \nabla\cdot(n_e \boldsymbol{V}_e) = 0, \qquad (2)$$

$$mn_e(\partial_t + \boldsymbol{V}_e\cdot\nabla)\boldsymbol{V}_e = -en_e(\boldsymbol{E} + \boldsymbol{V}_e\times\boldsymbol{B}) - \nabla\cdot\boldsymbol{\Pi}_e - \nabla P_e + \boldsymbol{\mathcal{C}}_{ei} + \boldsymbol{F}_Q, \qquad (3)$$

and

$$n_e(\partial_t + \boldsymbol{V}_e\cdot\nabla)\boldsymbol{S} = \frac{2\mu_B n_e}{\hbar}\boldsymbol{B}\times\boldsymbol{S} - \nabla\cdot\mathbf{K} + \boldsymbol{\Omega}_S, \qquad (4)$$

respectively. Here we have added the electron-ion collisions $\boldsymbol{\mathcal{C}}_{ei}$, denoted the total quantum force density by

$$\boldsymbol{F}_Q = -n_e\langle\nabla Q_{(\alpha)}\rangle - \frac{2\mu_B n_e}{\hbar}(\nabla\otimes\boldsymbol{B})\cdot\boldsymbol{S} - \frac{1}{m_e}\nabla\cdot(n_e\boldsymbol{\Sigma}) - \frac{1}{m_e}\nabla\cdot(n_e\tilde{\boldsymbol{\Sigma}}) - \frac{1}{m_e}\nabla\cdot(n_e\hat{\boldsymbol{\Sigma}}), \qquad (5)$$

where $\hat{\boldsymbol{\Sigma}} = 2\mathrm{Sym}[(\nabla S_a)\otimes\langle\nabla\mathcal{S}^a_{(\alpha)}\rangle]$ denotes the symmetric part of the tensor, and defined the nonlinear spin fluid contribution by

$$\begin{aligned}\boldsymbol{\Omega}_S = {}& \frac{1}{m_e}\boldsymbol{S}\times[\partial_a(n_e\partial^a\boldsymbol{S})] + \frac{1}{m_e}\boldsymbol{S}\times[\partial_a(n_e\langle\partial^a\boldsymbol{\mathcal{S}}_{(\alpha)}\rangle)] \\ &+ \frac{n_e}{m_e}\left\langle\frac{\boldsymbol{\mathcal{S}}_{(\alpha)}}{n_{(\alpha)}}\times[\partial_a(n_{(\alpha)}\right\rangle\partial^a\boldsymbol{S})] \\ &+ \frac{n_e}{m_e}\left\langle\frac{\boldsymbol{\mathcal{S}}_{(\alpha)}}{n_{(\alpha)}}\times[\partial_a(n_{(\alpha)}\partial^a\boldsymbol{\mathcal{S}}_{(\alpha)})]\right\rangle,\end{aligned} \qquad (6)$$

where $\boldsymbol{\Pi}_e = m_e n_e[\langle\boldsymbol{w}_{(\alpha)}\otimes\boldsymbol{w}_{(\alpha)}\rangle - \mathbf{I}\langle w^2_{(\alpha)}\rangle/3]$ is the trace-free anisotropic pressure tensor ($\mathbf{I}$ is the unit tensor), $P_e = m_e n_e\langle w^2_{(\alpha)}\rangle$ is the isotropic scalar pressure, $\boldsymbol{\Sigma} = (\nabla S_a)\otimes(\nabla S^a)$ is the nonlinear spin correction to the classical momentum equation, $\tilde{\boldsymbol{\Sigma}} = \langle(\nabla\mathcal{S}_{(\alpha)a})\otimes(\nabla\mathcal{S}^a_{(\alpha)})\rangle$ is a pressurelike spin term (which may be decomposed into a trace-free part and a trace), $\mathbf{K} = n_e\langle\boldsymbol{w}_{(\alpha)}\otimes\boldsymbol{\mathcal{S}}_{(\alpha)}\rangle$ is the thermal-spin coupling, and $[(\nabla\otimes\boldsymbol{B})\cdot\boldsymbol{S}]^a = (\partial^a B_b)S^b$. Here the latin indices $a, b, \ldots = 1, 2, 3$ denote the vector components. We note that the momentum conservation equation (3) and the spin evolution equation (4) still contain the explicit sum over the N states, and (as in classical fluid theory) it is necessary to impose further statistical relations in order to close the system [38]. The preceding analysis applies equally well to electrons as holes or similar condensations. We will now include the ion species, which, due to the smaller charge-to-mass ratio, are described by the classical equations of motion.

The coupling between the quantum plasma species is mediated by the electromagnetic field. By definition, we let $\boldsymbol{B}_{\mathrm{tot}}$ include spin sources, i.e., $\boldsymbol{B}_{\mathrm{tot}} \equiv \boldsymbol{B} + \boldsymbol{B}_{\mathrm{sp}}$, such that Ampere's law in terms of $\boldsymbol{B}_{\mathrm{tot}}$ reads $\nabla\times\boldsymbol{B}_{\mathrm{tot}} = \mu_0(\boldsymbol{j} + \boldsymbol{j}_{\mathrm{sp}}) + c^{-2}\partial_t\boldsymbol{E}$, including the magnetization spin current $\boldsymbol{j}_{\mathrm{sp}} = \nabla\times(2n\mu_B\boldsymbol{S}/\hbar)$ [39]. We obtain consistency with the momentum conservation equation (3) by adding a term proportional to $\boldsymbol{V}\times\boldsymbol{B}_{\mathrm{sp}}$ to the Lorentz force and subtracting it from the quantum force. The above alterations are only a reshuffling of terms. However, a difference does appear when closing the system using Faraday's law. By letting $\nabla\times\boldsymbol{E} = -\partial_t\boldsymbol{B}_{\mathrm{tot}}$, using $\boldsymbol{B}_{\mathrm{tot}}$ instead of $\boldsymbol{B}$, we indeed obtain a difference compared to the classical Maxwell's equations. It is the full electromagnetic fields, including spin sources, that should be used in Faraday's law. Thus, Faraday's law as presented here is therefore the correct one to use. This form also gives a Hermitian susceptibility tensor (see below).

To demonstrate the usefulness of the spin fluid equations, we investigate linear wave propagation in a magnetized plasma. For comparison, we first neglect all quantum effects. Linearizing, Fourier analyzing the equations of motion, and substituting the velocities into Maxwells equations, we obtain $\boldsymbol{\varepsilon}\cdot\boldsymbol{E} = 0$, where $\boldsymbol{\varepsilon} = \mathbf{I} + \boldsymbol{\xi}$, with

$$\boldsymbol{\xi} = \begin{pmatrix} \chi_{\perp\perp} & \chi_{\perp\top} & \chi_{\perp z} \\ -\chi_{\perp\top} & \chi_{\top\top} & \chi_{\top z} \\ \chi_{\perp z} & -\chi_{\top z} & \chi_{zz}\end{pmatrix} + \begin{pmatrix} \frac{k_z^2c^2}{\omega^2} & 0 & \frac{k_zk_\perp c^2}{\omega^2} \\ 0 & \frac{k^2c^2}{\omega^2} & 0 \\ \frac{k_zk_\perp c^2}{\omega^2} & 0 & \frac{k_\perp^2c^2}{\omega^2}\end{pmatrix}, \qquad (7)$$

and the standard susceptibility components are

$$\begin{aligned}
\chi_{\perp\perp} &= -\sum_{\mathrm{p.s.}}\frac{\omega_p^2(\omega^2 - k_z^2v_t^2)}{\omega_w^4}, \\
\chi_{\perp\top} &= -i\sum_{\mathrm{p.s.}}\frac{\omega_p^2\omega_c(\omega^2 - k_z^2v_t^2)}{\omega\omega_w^4}, \\
\chi_{\perp z} &= -\sum_{\mathrm{p.s.}}\frac{\omega_p^2k_\perp k_zv_t^2}{\omega_w^4}, \\
\chi_{\top\top} &= -\sum_{\mathrm{p.s.}}\frac{\omega_p^2(\omega^2 - k^2v_t^2)}{\omega_w^4}, \\
\chi_{\top z} &= i\sum_{\mathrm{p.s.}}\frac{\omega_p^2\omega_ck_\perp k_zv_t^2}{\omega\omega_w^4}, \\
\chi_{zz} &= -\sum_{\mathrm{p.s.}}\frac{\omega_p^2(\omega^2 - \omega_c^2 - k_\perp^2v_t^2)}{\omega_w^4}.
\end{aligned} \qquad (8)$$

Here the sums are over the particle species, $k = (k_z^2 + k_\perp^2)^{1/2}$, $\boldsymbol{k}_\perp$ is the perpendicular (to $\hat{z}$) part of the wave vector, the $\top$ direction is parallel to $\hat{z} \times \boldsymbol{k}_\perp$, ω_p is the plasma frequency (ω_{pe} for the electrons and ω_{pi} for the ions), $\omega_c = qB_0/m$ is the cyclotron frequency, q and m are the particle charge and mass, respectively, v_t^2 is the square of the thermal velocity times the ratio of specific heats, c is the speed of light in vacuum, and $\omega_w^4 = \omega^2(\omega^2 - k^2 v_t^2) - \omega_c^2(\omega^2 - k_z^2 v_t^2)$. For notational convenience, the subscripts denoting the various particle species have been left out.

Next we determine the equilibrium spin configuration. For many plasmas, paramagnetic theory applies. Thus, in an external magnetic field $\boldsymbol{B}_0 = B_0\hat{z}$, the zero order magnetization $\boldsymbol{M}_{S0}$ due to the spin can be written [40] $\boldsymbol{M}_{S0} = n_0\mu_B\eta(\mu_B B_0/KT)\hat{z}$, where K is Boltzmann's constant, T is the temperature, and we have introduced the Langevin function $\eta(x) = [\coth(x) - x^{-1}]$. Here we have assumed that the spin contribution to the total magnetic field is small; otherwise, $B_0 \rightarrow B_0 + B_{S0}$, where $B_{S0} = \mu_0\mu_B n_0$. In general, the spin magnetization $\boldsymbol{M}_S$ and the spin vector $\boldsymbol{S}$ are related by $\boldsymbol{S} = \hbar\boldsymbol{M}_S/2n\mu_B$, and, thus, the zero order spin vector becomes $\boldsymbol{S}_0 = (\hbar/2)\eta(\mu_B B_0/KT)\hat{z}$. We then obtain the spin-current contribution $\boldsymbol{j}_s = \nabla \times (4n_e e\boldsymbol{S}/m_e)$.

Generalizing (7) to include all terms from quantum effects gives extremely complicated expressions. However, for most plasmas, the parameter $\mu_B B_0/KT$ is very small, the spins are essentially randomly oriented, and the spin quantum effects are negligible. On the other hand, for low-frequency wave motion in a highly magnetized (or low-temperature) plasma, the spin effects can be appreciable. In this case, the dominant contribution to the spin effects comes from the component of the spin force parallel to the magnetic field, $F_{Qz} = -(2\mu_B n_0 S_0/\hbar)\partial_z B_1$, where B_1 denotes the magnetic field perturbation, together with the part of the spin current in the $\top$ direction (from the part proportional to $\nabla n \times \boldsymbol{S}_0$), and we can drop all other components as well as quantum terms that are proportional to $\hbar^2$, provided $eB_0 \gg \hbar k^2$. Keeping the above terms, including only the lowest order contributions in ω/ω_{ci}, the susceptibility tensor is modified to

$$\boldsymbol{\chi} = \begin{pmatrix} \chi_{\perp\perp} & \chi_{\perp\top} & \chi_{\perp z} \\ -\chi_{\perp\top} & \chi_{\top\top} & \chi_{\top z} + \chi_{sp} \\ \chi_{\perp z} & -(\chi_{\top z} + \chi_{sp}) & \chi_{zz} \end{pmatrix}, \qquad (9)$$

where the spin contribution is

$$\chi_{sp} = i\eta\left(\frac{\mu_B B_0}{KT}\right)\frac{\omega_{p_e}^2 \hbar k_\perp k_z}{\omega(\omega^2 - k_z^2 v_{te}^2)m_e}. \qquad (10)$$

As an example, we consider the fast and slow magnetosonic modes, which are now described by the dispersion relation $\varepsilon_{\top\top}\varepsilon_{zz} + (\varepsilon_{\top z} + \varepsilon_{sp})^2 = 0$. For $\omega_{ci}^2/\omega_{pi}^2 \ll 1$, $\omega \ll k_z v_{te}$, the dispersion relation becomes

$$(\omega^2 - k^2 c_A^2)(\omega^2 - k_z^2 c_s^2) = \omega^2 k_\perp^2 c_s^2\left[1 + \eta\left(\frac{\mu_B B_0}{KT}\right)\frac{\hbar\omega_{ce}}{m_e v_{te}^2}\right]^2, \qquad (11)$$

where the ion-acoustic velocity is $c_s = (m_e/m_i)^{1/2} v_{te}$, the Alfvén velocity is $c_A = (B_0^2/\mu_0 n_0 m_i)^{1/2}$, and, for simplicity, we have assumed that the ion temperature is smaller than the electron temperature and included only electron thermal effects. Noting that $\mu_B B_0/KT \equiv \hbar\omega_{ce}/m_e v_{te}^2$, obviously the spin effects are important if $\hbar\omega_{ce}/m_e v_{te}^2 \gtrsim 1$. Thus, for laboratory magnetic fields, where at most $B_0 \sim$ 10–20 T, we need low-temperature plasmas for spin effects to influence the fast and slow magnetosonic modes.

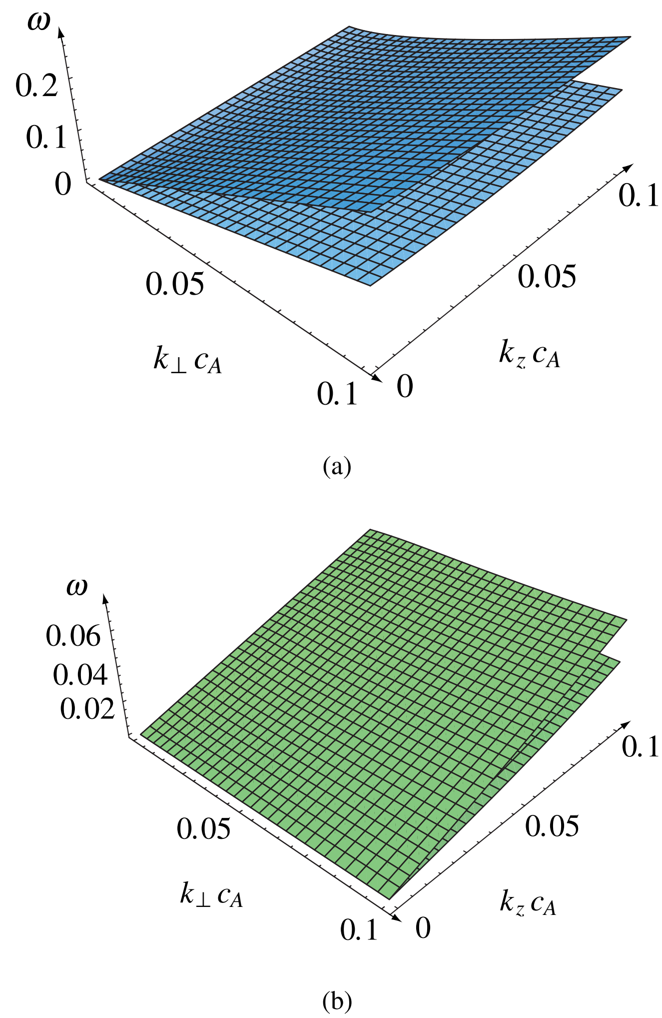

FIG. 1 (color online). The two roots of the dispersion relation (11) plotted in (a) fast mode and (b) slow mode. In case (a), the lower surface is without spin and the upper surface is with spin, while in (b), the lower surface is with spin and the upper surface is without spin. We note that the contribution from the spin term can be significant, in particular, for large values of the wave numbers k_z and $k_\perp$. Here we have used $c_s^2/c_A^2 = 0.5$, $\eta\hbar\omega_{ce}/m_e v_{te}^2 = 2$, and normalized the frequency by the ion cyclotron frequency ω_{ci} and the wave numbers by ω_{ci}/c_A.

However, in the vicinity of pulsars and magnetars [33], we have $B_0 \gtrsim 10^8$ T. For such systems, spin plasma effects can be important even in a high-temperature plasma. The spin effect on the fast and slow modes is illustrated in Fig. 1. Furthermore, we point out that, for modes with even lower phase velocities (which exist in, for example, dusty plasmas [16]), the relative importance of the spin susceptibility term is enhanced, and spin effects can be significant also under laboratory conditions.

In conclusion, we have derived the multifluid equations for spin-$\frac{1}{2}$ quantum plasmas, starting from the Pauli equation. In order to demonstrate the usefulness of our equations, we have analyzed the linear modes and demonstrated that the low-frequency modes are significantly altered by the spin effects provided that the condition $\hbar\omega_{ce}/m_e v_{te}^2 \gtrsim 1$ is fulfilled. In many classical plasmas, spin effects can be neglected due to the random orientations of the spin vector. We stress here, however, that our results show that the spin multifluid equations can have important applications to such different media as low-temperature solid state plasmas, as well as to the accretion disks surrounding pulsars and magnetars. Furthermore, we emphasize that the spin contributions are typically more important than the usual quantum plasma corrections [18], specifically when the inequality $eB_0 \gg \hbar k^2$ is fulfilled.

The linearized results presented in this Letter will most likely find experimental application in dusty plasmas, where the low phase velocity will make the relative importance of the spin contribution (10) particularly significant, enabling probing of the collective spin dynamics.

Finally, we suggest that the full nonlinear system (2)–(6) will show interesting behavior close to the electron cyclotron frequency, when the spin-vector evolution becomes resonant. Moreover, the importance of the pressurelike spin terms for, e.g., astrophysical plasmas is a further field of investigation.

[1] F. Haas, G. Manfredi, and M. R. Feix, Phys. Rev. E **62**, 2763 (2000).
[2] D. Anderson, B. Hall, M. Lisak, and M. Marklund, Phys. Rev. E **65**, 046417 (2002).
[3] F. Haas, L. G. Garcia, J. Goedert, and G. Manfredi, Phys. Plasmas **10**, 3858 (2003).
[4] F. Haas, Phys. Plasmas **12**, 062117 (2005).
[5] L. G. Garcia, F. Haas, L. P. L. de Oliviera, and J. Goedert, Phys. Plasmas **12**, 012302 (2005).
[6] M. Marklund, Phys. Plasmas **12**, 082110 (2005).
[7] M. Marklund and P. K. Shukla, Rev. Mod. Phys. **78**, 591 (2006).
[8] P. K. Shukla and B. Eliasson, Phys. Rev. Lett. **96**, 245001 (2006).
[9] P. A. Markowich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer, Vienna, 1990).
[10] F. Calvayrac, P.-G. Reinhard, E. Suraud, and C. Ullrich, Phys. Rep. **337**, 493 (2000).
[11] L. Stenflo, P. K. Shukla, and M. Marklund, Europhys. Lett. **74**, 844 (2006).
[12] K. H. Becker, K. H. Schoenbach, and J. G. Eden, J. Phys. D **39**, R55 (2006).
[13] Y. I. Salamin, S. X. Hu, K. Z. Hatsagortsyan, and C. H. Keitel, Phys. Rep. **427**, 41 (2006).
[14] G. A. Mourou, T. Tajima, and S. V. Bulanov, Rev. Mod. Phys. **78**, 309 (2006).
[15] P. K. Shukla and L. Stenflo, Phys. Lett. A **355**, 378 (2006).
[16] P. K. Shukla and A. A. Mamun, *Introduction to Dusty Plasma Physics* (Institute of Physics, Bristol, 2002).
[17] P. K. Shukla, Phys. Lett. A **352**, 242 (2006).
[18] G. Manfredi, quant-ph/0505004.
[19] B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).
[20] F. A. Blum, Phys. Rev. B **3**, 2258 (1971).
[21] A. V. Balatsky, Phys. Rev. B **42**, 8103 (1990).
[22] U. W. Rathe, C. H. Keitel, M. Protopapas, and P. L. Knight, J. Phys. B **30**, L531 (1997).
[23] S. X. Hu and C. H. Keitel, Phys. Rev. Lett. **83**, 4709 (1999).
[24] R. Arvieu, P. Rozmej, and M. Turek, Phys. Rev. A **62**, 022514 (2000).
[25] J. R. Vázquez de Aldana and L. Roso, J. Phys. B **33**, 3701 (2000).
[26] M. W. Walser and C. H. Keitel, J. Phys. B **33**, L221 (2000).
[27] M. W. Walser, D. J. Urbach, K. Z. Hatsagortsyan, S. X. Hu, and C. H. Keitel, Phys. Rev. A **65**, 043410 (2002).
[28] Z. Qian and G. Vignale, Phys. Rev. Lett. **88**, 056404 (2002).
[29] J. S. Roman, L. Roso, and L. Plaja, J. Phys. B **37**, 435 (2004).
[30] R. L. Liboff, Europhys. Lett. **68**, 577 (2004).
[31] J. N. Fuchs, D. M. Gangardt, T. Keilman, and G. V. Shlyapnikov, Phys. Rev. Lett. **95**, 150402 (2005).
[32] D. B. Melrose and A. J. Parle, Aust. J. Phys. **36**, 755 (1983); D. B. Melrose, *ibid.* **36**, 775 (1983); D. B. Melrose and A. J. Parle, *ibid.* **36**, 799 (1983).
[33] A. K. Harding and D. Lai, Rep. Prog. Phys. **69**, 2631 (2006).
[34] D. B. Melrose and J. I. Weise, Phys. Plasmas **9**, 4473 (2002).
[35] M. G. Baring, P. L. Gonthier, and A. K. Harding, Astrophys. J. **630**, 430 (2005).
[36] G. Brodin, M. Marklund, L. Stenflo, and P. K. Shukla, New J. Phys. **8**, 16 (2006).
[37] P. R. Holland, *The Quantum Theory of Motion* (Cambridge University Press, Cambridge, England, 1993).
[38] Using $L \gg \lambda_F$, where L is the typical fluid length scale and λ_F is the Fermi wavelength, we obtain [18] $\langle \nabla Q_{(\alpha)} \rangle \approx \nabla[-(\hbar^2/2m_e n_e^{1/2})\nabla^2 n_e^{1/2}]$. In the Schrödinger treatment of quantum plasmas, this term is the only quantum contribution to the equations of motion (see [1–6,18]).
[39] We note that this reshuffling of terms stems from the use of a nonrelativistic particle theory. Thus, it is expected that the need for such rearrangements will disappear in a fully relativistic theory.
[40] The full magnetization is known to include both Pauli spin paramagnetism and Landau orbit diamagnetism. However, here only the Pauli contribution should be included.