Viscous Plasma Evolution from Gravity Using Anti–de Sitter/Conformal-Field-Theory Correspondence

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We analyze the anti–de Sitter/conformal-field-theory dual geometry of an expanding boost-invariant plasma. We show that the requirement of nonsingularity of the dual geometry for leading and subasymptotic times predicts, without any further assumptions about gauge theory dynamics, hydrodynamic expansion of the plasma with viscosity coefficient exactly matching the one obtained earlier in the static case by Policastro, Son, and Starinets.

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One of the more challenging problems in theoretical physics is the understanding from first principles of the behavior of quark-gluon plasma, the phase of matter consisting of deconfined quarks and gluons. There are strong indications that the plasma observed at Relativistic Heavy Ion Collider (RHIC) is indeed strongly coupled (see, e.g., [1]) and is well described by models based on hydrodynamics [2]. This suggests the need for nonperturbative techniques to study the dynamics of the system. This need is especially acute if one would like to study nonequilibrium phenomena, thermalization, etc.

A very powerful technique for studying nonperturbative properties of gauge theory has emerged from string theory in recent years [3]. In its original form, the AdS/CFT correspondence states the equivalence of $\mathcal{N} = 4$ Super Yang-Mills theory and string theory in a curved 10-dimensional $AdS_5 \times S^5$ background. What is crucial is that the string theory is easiest to handle in the regime of strong gauge theory coupling.

Although the correspondence does not have a direct counterpart which works for QCD, it has been argued that for studying features of the plasma not too far above the deconfinement phase transition, it may be a good approximation as the plasma is strongly coupled and deconfined.

From a more general perspective, it is also interesting to study, for their own sake, dynamical time-dependent processes in the $\mathcal{N} = 4$ supersymmetric theory within the context of the AdS/CFT correspondence in order to have an example where such phenomenae can be calculated exactly, as well as to develop in this context new methods to address these dynamical issues. The methods used in the present Letter translate questions on the behavior of the plasma into certain questions within general relativity, and we hope that this may inspire further research in both domains.

Properties of the $\mathcal{N} = 4$ gauge theory at fixed finite temperature have been studied in some detail [4,5];, in particular, shear viscosity has been calculated from perturbations around a static black hole background. On a more qualitative level, thermalization has been suggested to correspond to black hole formation in the dual description [6], while cooling was advocated to correspond to black hole motion in the 5th direction [7].

In [8], a quantitative framework for studying such time-dependent phenomena has been proposed. The criterion of a nonsingular dual geometry was shown to pick out uniquely, in a boost-invariant setting, asymptotic perfect fluid hydrodynamical evolution for large proper-times. The resulting asymptotic geometry was shown to be analogous to a moving black hole. Further work within this framework includes [9,10].

The aim of this Letter is to show that the criterion of nonsingularity predicts, when applied also to subasymptotic times, viscous hydrodynamic evolution with a specific viscosity coefficient. As a by-product, we obtain a nontrivial consistency check of the AdS/CFT correspondence for the value of the shear viscosity.

Boost-invariant viscous hydrodynamics.—Let us consider the spacetime evolution of the energy-momentum tensor of an expanding fireball of plasma. The energy-momentum tensor is constrained by energy-momentum conservation

$$\partial_\mu T^{\mu\nu} = 0 \tag{1}$$

and, for the case of $\mathcal{N} = 4$ SYM theory that we consider here, tracelessness $T^\mu_\mu = 0$. We will further restrict ourselves to boost-invariant evolution, first considered by Bjorken [11] as a model of the midrapidity region in heavy-ion collisions. This assumption is also commonly used in hydrodynamic simulations for RHIC [2]. We will further assume no dependence on transverse coordinates. It is natural to use the proper-time or spacetime rapidity coordinates for Minkowski space:

$$ds^2 = -d\tau^2 + \tau^2 dy^2 + dx_\perp^2 \tag{2}$$

Then energy-conservation and tracelessnes determine $T^{\mu\nu}$ in terms of a single function—the energy density $\varepsilon(\tau)$ (for explicit expressions [12] see [8]). The dynamics of gauge theory should then determine $\varepsilon(\tau)$. In [8], we proposed to use AdS/CFT to determine the proper-time dependence of energy density by first constructing the dual geometry to a given $\varepsilon(\tau)$ and then requiring its nonsingularity to fix the physical $\varepsilon(\tau)$.

Let us review what would be the physical expectations for large proper-times if the gauge theory dynamics were described by viscous hydrodynamics. If there would be no viscosity, then we would have

$$\varepsilon(\tau) = \frac{1}{\tau^{4/3}}. \tag{3}$$

If in addition we would have viscosity [13]

$$\eta = \frac{\eta_0}{\tau}, \tag{4}$$

then the energy density would behave like (see [10])

$$\varepsilon(\tau) = \frac{1}{\tau^{4/3}} - \frac{2\eta_0}{\tau^2} + \ldots \tag{5}$$

with viscosity effects generating subleading deviations from the ideal fluid case (3).

Construction of a dual AdS/CFT geometry.—The procedure of constructing the dual geometry to a gauge theory configuration with given expectation value of the energy-momentum tensor was introduced in [14]. One adopts the Fefferman-Graham coordinates [15] for the 5-dimensional metric

$$ds^2 = \frac{\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dz^2}{z^2} \tag{6}$$

where the z coordinate is the "fifth" coordinate while μ is a 4D index. One solves Einstein equations with negative cosmological constant:

$$E_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - 6g_{\alpha\beta} = 0 \tag{7}$$

with a boundary condition for $\tilde{g}_{\mu\nu}$ around $z = 0$:

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + z^4\tilde{g}^{(4)}_{\mu\nu} + \ldots \tag{8}$$

The fourth order term is related to the expectation value of the energy-momentum tensor through

$$\langle T_{\mu\nu}\rangle = \frac{N_c^2}{2\pi^2}\tilde{g}^{(4)}_{\mu\nu} \tag{9}$$

The procedure is therefore to solve the 5-dimensional Einstein's equations (with negative cosmological constant $\Lambda = -6$) with the boundary condition (8) for a given spacetime profile of the gauge-theoretical $\langle T_{\mu\nu}\rangle$. In the following, since we will be dealing directly with the metric and hence with $\tilde{g}^{(4)}_{\mu\nu}$, so we will suppress the factor $N_c^2/(2\pi^2)$ throughout the computation, reinstating it only in the final discussion.

Nonsingularity and viscosity.—The metric consistent with the symmetries of the boost-invariant expansion has the form

$$ds^2 = \frac{-e^{a(\tau,z)}d\tau^2 + \tau^2 e^{b(\tau,z)}dy^2 + e^{c(\tau,z)}dx_\perp^2 + dz^2}{z^2} \tag{10}$$

In [8], the dual geometry was determined for asymptotic times with the energy-density behaving like

$$\varepsilon(\tau) = \frac{1}{\tau^s} \tag{11}$$

for large proper-times. The resulting metric coefficients were given for large proper-times as functions of the scaling variable

$$v = \frac{z}{\tau^{s/4}} \tag{12}$$

It was found in [8] that the resulting geometry was nonsingular only when $s = 4/3$, thus corresponding to perfect fluid hydrodynamics. Of course, subleading corrections like (5) are possible. The explicit leading coefficients for this case were found to be

$$\begin{aligned} a(v) &= \log\frac{(1 - v^4/3)^2}{1 + v^4/3} \\ b(v) &= c(v) = \log(1 + v^4/3) \end{aligned} \tag{13}$$

The resulting geometry looks like a black hole whose horizon (in the Fefferman-Graham coordinates) moves in the fifth dimension as $z_0 = 3^{1/4}\tau^{1/3}$. This leads to the temperature

$$T = \frac{\sqrt{2}}{\pi}\frac{1}{z_0} = \frac{\sqrt{2}}{\pi 3^{1/4}}\tau^{-1/3}. \tag{14}$$

The coefficient $\sqrt{2}$ comes from the special form of the black hole metric in Fefferman-Graham coordinates. See [8] for a discussion.

In an interesting recent paper, Nakamura and Sin [10] determined the leading, subasymptotic in proper-time, corrections to the metric coefficients like

$$a(\tau, z) = a(v) + a_1(v)\frac{1}{\tau^{2/3}} + \ldots \tag{15}$$

which represent $\varepsilon(\tau)$ of the form (5). The coefficients which can be extracted from the results of [10] in a form convenient for proceeding to higher orders are

$$\begin{aligned} a_1(v) &= 2\eta_0\frac{(9 + v^4)v^4}{9 - v^8} \\ b_1(v) &= -2\eta_0\frac{v^4}{3 + v^4} + 2\eta_0\log\frac{3 - v^4}{3 + v^4} \\ c_1(v) &= -2\eta_0\frac{v^4}{3 + v^4} - \eta_0\log\frac{3 - v^4}{3 + v^4} \end{aligned} \tag{16}$$

However to this order in τ ($\mathcal{O}(\tau^{-2/3})$), the Riemann tensor squared

$$\mathfrak{R}^2 \equiv R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} \tag{17}$$

was found to be finite for any η_0, the first divergence appearing only at order $\tau^{-4/3}$. This behavior suggested that in order to determine the coefficient of viscosity η_0, one had to go one order higher in $\tau^{-2/3}$. Namely, one has to find the metric coefficients to second order like

$$a(\tau, z) = a(v) + a_1(v)\frac{1}{\tau^{2/3}} + a_2(v)\frac{1}{\tau^{4/3}} + \ldots \quad (18)$$

In order to setup a systematic expansion procedure, one expands the left hand side of the Einstein equations $E_A \equiv (\tau^{2/3}E_{\tau\tau}, \tau^{4/3}E_{\tau z}, \tau^{2/3}E_{zz}, \tau^{-4/3}E_{yy}, \tau^{2/3}E_{xx})$ in powers of $\tau^{-2/3}$:

$$E_A = E_A^{(0)}(v) + E_A^{(1)}(v)\frac{1}{\tau^{2/3}} + E_A^{(2)}(v)\frac{1}{\tau^{4/3}} + \ldots \quad (19)$$

where $A = 1\ldots5$ numbers the five nontrivial components of the Einstein equations mentioned above. The prefactors are chosen in such a way as to have a uniform expansion. The leading order results (13) satisfy $E_A^{(0)} = 0$ while the first subleading corrections (16) satisfy $E_A^{(1)} = 0$. Therefore, in order to find $a_2(v)$, $b_2(v)$, $c_2(v)$, we have to solve the equations $E_A^{(2)} = 0$ with the boundary condition that these coefficient functions should vanish at $v = 0$. The resulting expressions depend on the viscosity coefficient η_0 and another coefficient which we denote by C. The explicit expressions are

$$a_2(v) = \int_0^v \left(\frac{4w^5(27 + 9w^4 + 2w^8)}{3(9 - w^8)^2} - 24\eta_0^2\frac{w^3(405 + 171w^4 + 189w^8 + 5w^{12} - 2w^{16})}{(9 - w^8)^3} - C\frac{w^3(9 + 2w^4 + w^8)}{(9 - w^8)^2}\right)dw$$

$$b_2(v) = -2c_2(v) + \frac{v^2}{6 + 2v^4} - \frac{\operatorname{arctanh}\frac{v^2}{\sqrt{3}}}{2\sqrt{3}} + \eta_0^2\frac{v^4(39 + 7v^4)}{(3 + v^4)^2} + \frac{3}{2}\eta_0^2\log\frac{3 - v^4}{3 + v^4} + C\frac{v^4}{12(3 + v^4)} \quad (20)$$

$$c_2(v) = \int_0^v \left(\frac{4w^9(9 + w^4)}{3(9 - w^8)^2} + \frac{8w^3\operatorname{arctanh}\frac{w^2}{\sqrt{3}}}{\sqrt{3}(9 - w^8)} + \eta_0^2\frac{24w^3(w^4 - 15)(3 + 5w^4)}{(3 - w^4)^2(3 + w^4)^3} - C\frac{w^3(1 + w^4)}{(3 - w^4)(3 + w^4)^2}\right)dw$$

One can express these integrals in terms of elementary functions and dilogarithms, but the expressions are rather lengthy and will not be presented here.

It remains to determine when the background geometry given by the coefficients up to second order is nonsingular. To this end, we calculate the curvature invariant $\mathfrak{R}^2$ defined by (17) and expand it in the scaling limit up to the order $\tau^{-4/3}$. The resulting expression at this order has the form

$$\mathfrak{R}^2 = \text{nonsingular terms} + \frac{1}{\tau^{4/3}} \times \frac{\text{polynomial in } v,\ \eta_0 \text{and } C}{(3 - v^4)^4(3 + v^4)^6}. \quad (21)$$

We see that there is a potential singularity at $v = 3^{1/4}$. It turns out that the singularity is cancelled exactly when

$$\eta_0^2 = \frac{\sqrt{3}}{18}. \quad (22)$$

There is no restriction on C at this order. We expect one would have to perform the analysis to the next order to fix C [16].

The above calculation shows that nonsingular dual geometry is not possible for an exact perfect fluid, but that viscosity effects in the proper-time evolution are present with a uniquely fixed value of the viscosity coefficient given by (22). It is interesting to compare this value with the shear viscosity obtained by Policastro, Son, and Starinets [4] who derived it by studying the response of a static plasma at fixed temperature to small perturbations.

To this end, let us take the value of (shear) viscosity obtained in [4] at a fixed temperature T:

$$\eta = \frac{1}{4\pi}s = \frac{\pi}{8}N_c^2T^3 \quad (23)$$

If we insert the proper-time dependence of the temperature (14) for the evolving plasma into the above expression, we obtain

$$\eta = \frac{N_c^2}{2\pi^2}\frac{1}{2^{1/2}3^{3/4}}\frac{1}{\tau} \quad (24)$$

where we have factored out the coefficient $N_c^2/(2\pi^2)$ appearing in (9). The resulting estimate for the viscosity coefficient η_0 is therefore

$$\eta_0 = \frac{1}{2^{1/2}3^{3/4}} \equiv \left(\frac{\sqrt{3}}{18}\right)^{1/2} \quad (25)$$

which is exactly the value (22) for which the dual background geometry of the evolving plasma is nonsingular.

Discussion.—In this Letter, we have studied the proper-time evolution of a boost-invariant plasma using the AdS/CFT correspondence. We have shown that the requirement that the dual geometry is nonsingular predicts the proper-time evolution of the energy density to be equal [17] to the one found from viscous hydrodynamics with the viscosity being exactly the one following from $\eta/s = 1/(4\pi)$ in the static case. The computation involves the nonlinear regime of gravity on the AdS/CFT side, and it is encouraging for other possible applications that the method can capture such fine details of the gauge theory dynamics.

It would be very interesting to study in more detail the features of this geometry and its thermodynamics [16], as well as to apply these techniques to other dynamical nonequilibrium processes.

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