Vanishing Bulk Viscosities and Conformal Invariance of the Unitary Fermi Gas

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By requiring general-coordinate and conformal invariance of the hydrodynamic equations, we show that the unitary Fermi gas has zero bulk viscosity, $\zeta = 0$, in the normal phase. In the superfluid phase, two of the bulk viscosities have to vanish, $\zeta_1 = \zeta_2 = 0$, while the third one ζ_3 is allowed to be nonzero.

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The unitary Fermi gas is a system where fermions of two components interact through a zero-range two-body potential fine-tuned to infinite scattering length. This system is under active study, both experimentally [1–4] and theoretically. At zero temperature, the system is located midway between the Bose-Einstein condensation (BEC) and the BCS regimes [5–7]. The most remarkable property of this system is the absence of any intrinsic dimensionful parameter except the density and the temperature. This implies, on the one hand, a lack of any small perturbative parameter. On the other hand, the properties of the system are universal, i.e., independent of the microscopic details, which leads to great simplification in many problems.

Several recent experiments have concentrated on dynamic properties of the unitary Fermi gas. The trap breathing mode was studied in Ref. [8–11], in which it was observed that the damping rate reaches a minimum near unitary. The expansion of the gas as it is released from a trap was studied in Refs. [12,13], where a hydrodynamic behavior was observed. Such dynamic processes should depend, to one or another degree, on the kinetic coefficients. According to fluid mechanics [14], a normal gas has three kinetic coefficients: the shear viscosity η, the bulk viscosity ζ, and the thermal conductivity κ. This is also the situation for the Fermi gas above the critical temperature. Below the critical temperature, the Fermi gas is in the superfluid phase, and the number of kinetic coefficients is five [14,15]: η, κ, and three bulk viscosities, normally denoted as ζ_1, ζ_2, and ζ_3.

The purpose of this Letter is to show that at unitarity (i.e., when the scattering length is infinite), certain kinetic coefficients vanish exactly. Namely, in the normal phase the bulk viscosity vanishes, and in the superfluid phase two of three bulk viscosities, ζ_1 and ζ_2, vanish.

To emphasize the nontriviality of this result, we note that it is by no mean related to the well-known fact that the Boltzmann equation for a classical monoatomic gas, with two-body collision, implies zero bulk viscosity. When three-body collisions are included, the Boltzmann equation generically yields a nonzero bulk viscosity [16]. In our case, the bulk viscosities vanish even when the Boltzmann equation cannot be used, but the result is not expected to hold outside the unitarity regime.

In the unitarity regime, scale invariance has been used to derive nontrivial relationships between thermodynamic observables [17]. But scale invariance, *per se*, does not imply the vanishing of any kinetic coefficient. It only implies that all kinetic coefficients are homogeneous functions of the temperature T and the chemical potential μ. From scale invariance one can immediately write down the scaling behavior of the shear viscosity η, the bulk viscosity ζ, and the thermal conductivity κ,

$$\eta = \hbar n \tilde{\eta}\left(\frac{T}{\mu}\right), \qquad \zeta = \hbar n \tilde{\zeta}\left(\frac{T}{\mu}\right), \qquad \kappa = \frac{\hbar n}{m}\tilde{\kappa}\left(\frac{T}{\mu}\right), \tag{1}$$

where $\tilde{\eta}$, $\tilde{\zeta}$, and $\tilde{\kappa}$ are dimensionless functions of the ratio of the temperature and the chemical potential. In the superfluid phase, instead of one function $\tilde{\zeta}$ one has three functions $\tilde{\zeta}_1$, $\tilde{\zeta}_2$, $\tilde{\zeta}_3$.

The result advertised above is therefore stronger than what simple scaling arguments would imply. Heuristically, the vanishing of the bulk viscosities can be understood as follows. Consider a blob of a unitary Fermi gas that undergoes a uniform expansion, where the velocity $\mathbf{v}$ at point $\mathbf{x}$ being $\mathbf{v}(\mathbf{x}) = c\mathbf{x}$, where c is some constant. Because the unitary Fermi gas does not have any intrinsic scale, it should remains in thermal equilibrium throughout the whole process of uniform expansion. This means entropy is not produced during a flow with $\mathbf{v} = c\mathbf{x}$. Looking at the equation for entropy production, one finds $\zeta = 0$. Applying the same argument for a blob of a superfluid unitary Fermi gas undergoing uniform expansion with the same normal and superfluid velocity profile, $\mathbf{v}_n = \mathbf{v}_s = c\mathbf{x}$, we find $\zeta_2 = 0$. From the well-known inequality $\zeta_1^2 \le \zeta_2\zeta_3$ one concludes ζ_1 too has to vanish, and only ζ_3 can be nonzero.

Putting this argument on a more precise footing is, however, not straightforward. There is no regular solution to the hydrodynamic equations that describe uniform expansion. In the rest of this Letter we put the heuristic argument above on a firm mathematical ground. We will show how the vanishing of ζ in the normal phase and ζ_1 and ζ_2 follows from the requirement that the hydrodynamic equation exhibits the conformal invariance of the microscopic theory.

A general discussion of symmetries of the unitary Fermi gas is given in Ref. [18]; for convenience we briefly review these symmetries here. Let us start by discussing the microscopic theory. Because of the universality of the unitary Fermi gas, any short-range two-body interaction can be used, if it corresponds to infinite scattering length. In particular, we can choose to work with the following local Lagrangian (here and below $\hbar = 1$),

$$\mathcal{L} = i\psi^\dagger \partial_t \psi - \frac{1}{2m}|\nabla\psi|^2 + q_0 \psi^\dagger \psi \sigma - \frac{1}{2}(\nabla\sigma)^2 - \frac{\sigma^2}{2r_0^2}, \tag{2}$$

which describes a system of fermions interacting through the Yukawa potential $V(r) = -q_0^2(4\pi r)^{-1}e^{-r/r_0}$. Infinite scattering length is achieved by requiring $mq_0^2 r_0$ to be equal to a critical number (whose numerical value is 21.1...). Moreover, r_0 and q_0 can be time dependent; the only requirement is that $q_0^2 r_0$ is held fixed. For universality we also need to keep $r_0(t)$ small compared to any length scale in the problem. In particular, if one performs the following transformation

$$\begin{aligned} q_0(t) \to q_0'(t) &= \gamma(t) q_0(t), \\ r_0(t) \to r_0'(t) &= \gamma^{-2}(t) r_0(t), \end{aligned} \tag{3}$$

then, from the point of view of the physics at length scale larger than r_0, we have mapped our theory to itself, but not to another theory, provided that $mq_0^2 r_0$ is tuned to the value corresponding to infinite scattering length.

The best way to expose the symmetries of the system is to put it in a background gauge field A_μ and in a curved space with a 3D metric tensor $g_{ij}(t, \mathbf{x})$ [18]. The curved space here is a not physical; it is simply a trick to get physical results. The action is now

$$\begin{aligned} S = \int dt d\mathbf{x} \sqrt{g} \bigg[&\frac{i}{2}\psi^\dagger \overleftrightarrow{\partial}_t \psi - \frac{g^{ij}}{2m}(\partial_i + iA_i)\psi^\dagger(\partial_j - iA_j)\psi \\ &+ (q_0\sigma - A_0)\psi^\dagger\psi - \frac{g^{ij}}{2}\partial_i\sigma\partial_j\sigma - \frac{\sigma^2}{2r_0^2}\bigg], \end{aligned} \tag{4}$$

where $g = \det|g_{ij}|$ and g^{ij} is the inverse metric of g_{ij}. The action is invariant under the following infinitesimal transformations: (i) gauge transformations, parametrized by a function of space and time $\alpha(t, \mathbf{x})$,

$$\delta\psi = i\alpha\psi, \qquad \delta A_0 = -\partial_t\alpha, \qquad \delta A_i = -\partial_i\alpha; \tag{5}$$

(ii) local diffeomorphism, parametrized by three functions of space and time $\xi^i(t, \mathbf{x})$, $i = 1, 2, 3$,

$$\delta\psi = -\xi^k\partial_k\psi, \qquad \delta\sigma = -\xi^k\partial_k\sigma, \tag{6a}$$

$$\delta A_0 = -\xi^k\partial_k A_0 - A_k\dot{\xi}^k, \tag{6b}$$

$$\delta A_i = -\xi^k\partial_k A_i - A_k\partial_i\xi^k + mg_{ik}\dot{\xi}^k, \tag{6c}$$

$$\delta g_{ij} = -\xi^k\partial_k g_{ij} - g_{ik}\partial_j\xi^k - g_{kj}\partial_i\xi^k, \tag{6d}$$

and (iii) "conformal transformations," parametrized by a function $\beta(t)$ of time only,

$$\delta O = -\beta\dot{O} - \Delta[O]\dot{\beta}O, \tag{7}$$

where $\Delta[O]$ is the dimension of the field O, defined so that $\Delta[\psi] = \Delta[\sigma] = \frac{3}{4}$, $\Delta[A_0] = 1$, $\Delta[A_i] = 0$, and $\Delta[g_{ij}] = -1$. One can think about (7) as time reparametrization $t \to t' = t + \beta$. By direct substitution one can check that the action (4) is invariant under the transformations (5) and (6), and under (7) combined with (3) with $\gamma = 1 - \frac{1}{4}\dot{\beta}$. Galilean invariance is a combination of (5) and (6) with $\alpha = m\mathbf{V}\cdot\mathbf{x}$ and $\xi^k = V^k t$, and scale invariance is a combination of (6) and (7) with $\beta = bt$ and $\xi^k = \frac{1}{2}bx^k$, where V^k and b are constants.

The local diffeomorphism (6) reduces to reparametrization of space when ξ^k is time independent: the transformation laws come directly from the 3D tensor structure of the object under consideration (i.e., scalar in the case of ψ, σ, and A_0, vector in the case of A_i, and tensor in the case of g_{ij}). Equations (6) extend this invariance to *time-dependent* diffeomorphisms.

In flat space $g_{ij} = \delta_{ij}$ and zero background fields $A_0 = A_i = 0$, the long-distance, long-time dynamics of the system is described by a set of hydrodynamic equations [14]. This should remain true when the external fields are turned on and when the space is curved, if the following conditions are met: the fields are sufficiently weak and vary over length and time scales much larger than all microscopic scales; the curvature of space is small and varies slowly, also compared to all microscopic scales. The hydrodynamic equation should then be properly modified to take into account the external field and the curved metric. In writing these equations down, we require that the symmetries of the microscopic theory are inherited by the hydrodynamic theory. This condition arises from the fact that the hydrodynamic equations, which can be used to determine the response of the system on external perturbations, imply concrete forms of the fully retarded Green's functions of the hydrodynamic variables. The Ward identities that come from the symmetries (6) should be satisfied by these fully retarded Green's functions, which is achieved if the hydrodynamic equations have the same symmetries [19].

Our strategy is the following. First we will modify the standard hydrodynamic equations for the case of a nonzero external gauge field and a curved metric. There is a unique way to do so, as we shall see. Then we will show that the resulting equations are inconsistent with the conformal invariance unless certain kinetic coefficients vanish.

Consider the normal phase first. We shall write the hydrodynamic equations in term of the local mass density ρ, the local velocity v^i, and the local entropy per unit mass s. The set consists of the continuity equation,

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\rho) + \nabla_i(\rho v^i) = 0, \tag{8}$$

the equation of momentum conservation,

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\rho v_i) + \nabla_k \Pi_i^k = \frac{\rho}{m}(E_i - F_{ik}v^k), \qquad (9)$$

and the entropy production equation

$$\frac{1}{\sqrt{g}}\partial_t(\sqrt{g}\rho s) + \nabla_i\left(\rho v^i \partial_i s - \frac{\kappa}{T}\partial^i T\right) = \frac{2R}{T}. \qquad (10)$$

We follow closely the notations of Ref. [14]: Π_{ik} is the stress tensor, κ is the thermal conductivity, and R is the dissipative function. The obvious modifications are the replacement of the derivatives ∂_i by the covariant derivatives ∇_i [20] and the appearance of the force term in the momentum conservation Eq. (9), which comes from the electric ($E_i = \partial_t A_i - \partial_i A_0$) force and the magnetic $F_{ik} = \partial_i A_k - \partial_k A_i$) Lorentz force. The stress tensor can be written as

$$\Pi_{ik} = \rho v_i v_k + p g_{ik} - \sigma'_{ik}, \qquad (11)$$

where p is the pressure and σ'_{ik} is the viscous stress tensor. The information about the kinetic coefficients is contained in σ'_{ik} and R.

Consider the dissipationless limit first, setting $\sigma' = R = 0$. One can check that the hydrodynamic equations are invariant with respect to the general-coordinate transformations, provided that A_0, A_i, and g_{ij} transform as in Eqs. (6), and ρ, s, and v^i transform as

$$\delta\rho = -\xi^k \partial_k \rho, \qquad \delta s = -\xi^k \partial_k s, \qquad (12)$$

$$\delta v^i = -\xi^k \partial_k v^i + v^k \partial_k \xi^i + \dot{\xi}^i. \qquad (13)$$

The transformation law for v^i contains terms coming from the vector nature of v^i, but also a $\dot{\xi}^i$ term that can be understood if one recalls $\mathbf{v}$ changes under Galilean boosts ($\xi^i = V^i t$): $\mathbf{v} \to \mathbf{v} + \mathbf{V}$.

Now consider the dissipative terms. To keep the equation consistent with diffeomorphism invariance, one must require that σ'_{ik} and R transform as a two-index tensor and a scalar, respectively,

$$\delta\sigma'_{ij} = -\xi^k \partial_k \sigma'_{ij} - \sigma_{kj}\partial_i \xi^k - \sigma_{ik}\partial_j \xi^k, \qquad (14)$$

$$\delta R = -\xi^k \partial_k R. \qquad (15)$$

In flat space the viscous stress tensor is given by

$$\sigma'_{ij} = \eta(\partial_i v_j + \partial_j v_i) + (\zeta - \tfrac{2}{3}\eta)\delta_{ij}\partial_k v^k. \qquad (16)$$

The naive extension to curve space,

$$\sigma'^{\text{naive}}_{ij} = \eta(\nabla_i v_j + \nabla_j v_i) + (\zeta - \tfrac{2}{3}\eta) g_{ij}\nabla_k v^k, \qquad (17)$$

is, however, not a pure two-index tensor: its variation under diffeomorphism contains extra terms proportional to $\dot{\xi}^k$:

$$\delta\sigma'^{\text{naive}}_{ij} = \eta[\nabla_i(g_{jk}\dot{\xi}^k) + \nabla_j(g_{ik}\dot{\xi}^k)] + (\zeta - \tfrac{2}{3}\eta) g_{ij}\nabla_k \dot{\xi}^k + [\text{terms in Eq. (14)}]. \qquad (18)$$

These terms can be canceled out by adding terms proportional to the derivatives of the metric tensor to σ'_{ik}. Limiting oneself to terms containing the least number of derivatives, the terms needed are determined uniquely,

$$\sigma'_{ij} = \eta(\nabla_i v_j + \nabla_j v_i + \dot{g}_{ij}) + \left(\zeta - \frac{2}{3}\eta\right) g_{ij}\left(\nabla_k v^k + \frac{\dot{g}}{2g}\right). \qquad (19)$$

Now one can check that σ'_{ik} transforms according to Eq. (14).

Similarly, the dissipative function R becomes, in curved space

$$2R = \frac{\eta}{2}\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3}g_{ij}\nabla_k v^k + \dot{g}_{ij} - \frac{1}{3}g_{ij}\frac{\dot{g}}{g}\right)^2 + \zeta\left(\nabla_i v_i + \frac{\dot{g}}{2g}\right)^2 + \frac{\kappa}{T}\partial_i T \partial^i T. \qquad (20)$$

Let us now specialize on Fermi systems at infinite scattering length, and discuss its conformal invariance. The dissipationless hydrodynamic equation is invariant under (7), if the dimensions of different fields are

$$\Delta[\rho] = 2\Delta[\psi] = \tfrac{3}{2}, \qquad \Delta[s] = 0, \qquad \Delta[v^i] = 1. \qquad (21)$$

Now let us consider the dissipation terms. From dimensional analysis one find that one has to set

$$\Delta[\eta] = \Delta[\zeta] = \Delta[\kappa] = \tfrac{3}{2} \qquad (22)$$

for the hydrodynamic equation to be scale invariant. However, conformal invariance is not preserved generically. The culprit is the $\dot{g}_{ij}$ that transforms as

$$\delta\dot{g}_{ij} = -\beta\ddot{g}_{ij} + \ddot{\beta} g_{ij}, \qquad (23)$$

which leads to σ'_{ij} and R not to conform to the pattern of (7),

$$\delta\sigma'_{ij} = -\beta\dot{\sigma}'_{ij} - \tfrac{3}{2}\dot{\beta}\sigma'_{ij} + \tfrac{3}{2}\zeta\ddot{\beta}g_{ij}, \qquad (24a)$$

$$\delta R = -\beta\dot{R} - \frac{7}{2}\dot{\beta}R + \frac{3}{2}\zeta\ddot{\beta}\left(\nabla_i v^i + \frac{\dot{g}}{2g}\right), \qquad (24b)$$

unless the bulk viscosity ζ vanishes. Thus the requirement of conformal invariance of the hydrodynamic equations implies $\zeta = 0$. [Had we known only the scale invariance, i.e., had we restricted $\beta(t)$ to be of the form $\beta(t) = bt$, the $\ddot{\beta}$ terms would have been absent from Eqs. (24) and we would have been unable to reach this conclusion.]

Similarly, we can repeat the argument for the superfluid case. The hydrodynamics of superfluids contains an additional degree of freedom, which is the condensate phase φ, whose gauge-covariant gradient is the superfluid velocity,

$$v_i^s = \frac{\hbar}{m}(\partial_i\varphi + A_i). \qquad (25)$$

It transforms in the same way as the normal velocity

$v_i \equiv v_i^n$ under general-coordinate and conformal transformations. A consequence is that the relative velocity between the superfluid and the normal component $w^i = v_s^i - v^i$ transforms as a pure vector under diffeomorphism,

$$\delta w^i = -\xi^k \partial_k w^i + w^k \partial_k \xi^i. \tag{26}$$

The $\dot{\xi}^i$ term in the variation cancels between δv_s and δv.

The diffeomorphism-invariant dissipative function in curved space is

$$\begin{aligned} 2R = {} & \frac{\eta}{2}\left(\nabla_i v_j + \nabla_j v_i - \frac{2}{3} g_{ij} \nabla_k v^k + \dot{g}_{ij} - \frac{1}{3} g_{ij} \frac{\dot{g}}{g}\right)^2 \\ & + 2\zeta_1\left(\nabla_i v^i + \frac{\dot{g}}{2g}\right)\nabla_j(\rho_s w^j) + \zeta_2\left(\nabla_i v^i + \frac{\dot{g}}{2g}\right)^2 \\ & + \zeta_3[\nabla_i(\rho_s w^i)]^2 + \frac{\kappa}{T}\partial_i T \partial^i T. \end{aligned} \tag{27}$$

Under conformal transformations, R transforms as

$$\delta R = -\beta \dot{R} - \frac{7}{2}\dot{\beta} R + \frac{3}{2}\zeta_1 \ddot{\beta} \nabla_i(\rho w^i) + \frac{3}{2}\zeta_2 \ddot{\beta}\left(\nabla_i v^i + \frac{\dot{g}}{2g}\right). \tag{28}$$

The requirement of conformal invariance of superfluid hydrodynamics implies that the $\ddot{\beta}$ terms must have vanishing coefficients, i.e., $\zeta_1 = \zeta_2 = 0$.

In conclusion, we find that in the unitary limit the bulk viscosity vanishes in the normal phase. In the superfluid phase two of the three bulk viscosities vanishes. This vanishing of the bulk viscosities is directly related to the conformal invariance of the unitary Fermi gas.

It should be possible to check the result derived in this Letter by using the Boltzmann equation in the two regimes where it applies: the high-temperature regime $T \gg \mu$ and the low temperature regime $T \ll \mu$. In the intermediate regime $T \sim \mu$ the Boltzmann equation is not reliable, since we do not have weakly coupled quasiparticles. The result derived in this Letter, however, should be valid for all regimes of T/μ.

With respect to the recent experimental findings [8–11] it is tempting to speculate that the dip in the damping rate of the radial breathing modes near the Feshbach resonance could *partially* be due to the vanishing of the bulk viscosities at unitarity. Another source for the reduction of damping is probably the decrease of the shear viscosity and the thermal conductivity at strong coupling. A study of the breathing modes using two-fluid dissipative hydrodynamics may enable one to extract kinetic coefficients from experimental data, and ultimately verify the results derived in this Letter.

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