Probability Distribution of the Maximum of a Smooth Temporal Signal

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We present an approximate calculation for the distribution of the maximum of a smooth stationary temporal signal $X(t)$. As an application, we compute the persistence exponent associated with the probability that the process remains below a nonzero level M. When $X(t)$ is a Gaussian process, our results are expressed explicitly in terms of the two-time correlation function, $f(t) = \langle X(0)X(t)\rangle$.

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The problem of evaluating the distribution of the maximum of a time-correlated random variable $X(t)$ has elicited a large body of work by mathematicians [1–3] and physicists, both theorists [4–12] and experimentalists [13–17]. In the physics literature, this is related to the persistence problem, the probability that a temporal signal X (and hence its maximum) remains below a given level M up to time t. Persistence properties have been measured in as different systems as breath figures [13], liquid crystals [14], laser-polarized Xe gas [15], fluctuating steps on a Si surface [16], or soap bubbles [17].

The mathematical literature has mainly focused on evaluating $P_<(t) = \mathrm{Prob}(X(t') < M, t' \in [0, t])$ for Gaussian processes and for large $|M|$, a regime where efficient bounds or equivalent have been obtained [1,2]. Recently [3], and for Gaussian processes only, a numerical method to obtain valuable bounds has been extended to all values of M, although the required numerical effort can become quite considerable for large t.

Physicists have also concentrated their attention on Gaussian processes [7–10], which are often a good or *exact* description of actual physical processes. For instance, the total magnetization in a spin system [11], or the height profile of certain fluctuating interfaces [12,15,16] are true temporal Gaussian processes. Two general methods have been developed, focusing on the case $M = 0$, which applies to many physical situations. The first one [7–9] is a perturbation of the considered process around the Markovian Gaussian process, which has been extended for small values of M [8]. Within this method, only the large time asymptotics of $P_<(t)$ is known, leading to the definition of the persistence exponent (see below). The alternative method, using the independent interval approximation [10], gives very accurate results for smooth processes, but is restricted to $M = 0$.

In addition, this problem has obvious applications in many other applied and experimental sciences, where one has to deal with data analysis of complex statistical signals. For instance, statistical bounds of noisy signals are extremely useful for image processing (for instance in medical imaging or astrophysics [18]), in order to obtain cleaner images by correcting spurious bright or dark pixels [1,3]. In general, it is important to be able to evaluate the maximum of a correlated temporal or spatial signal originating from experimental noise. The same question can arise when the signal lives in a more abstract space. For instance, in the context of genetic cartography, statistical methods to evaluate the maximum of a complex signal has been exploited to identify putative quantitative trait loci [19]. Finally, this same problem arises in econophysics or finance, where the probability for a generally strongly correlated financial signal to remain below or above a certain level is of great concern.

One considers a *general* (i.e., not necessarily Gaussian) stationary process $X(t)$ of distribution $g(X)$, and zero mean. The process is assumed to be "smooth", so that its velocity $X'(t)$ is continuous, ensuring that the number of times $N(t)$, where $X = M$, remains finite for any bounded time interval $[0, t]$. For a given level M, one defines τ as the average temporal interval between two crossings of the level $X = M$. One also introduces $P_-(t)$ $[P_+(t)]$ as the distribution of time intervals during which $X(t) \le M [X(t) \ge M]$. The average of $P_\pm(t)$ are denoted by $\tau_\pm$. Finally, one defines $P_<(t)$ $[P_>(t)]$ as the probability, starting at $X(0) < M$ $[X(0) > M]$, that the process X remains below the level M (above the level M) up to time t. The distribution of the maximum (minimum) of the process $X(t)$ in the interval $[0, t]$ is clearly the derivative of $P_<(t)$ $[P_>(t)]$ with respect to M. The difficulty of obtaining analytic forms for the above quantities lies in the fact that powerful methods like the Fokker-Planck approach are useless for non-Markovian processes. In this Letter, we obtain closed expressions for $P_<(t)$, $P_>(t)$, and $P_\pm(t)$ from a minimal knowledge of the statistical properties of the process X. This is achieved by analyzing the trajectories of X and using the sole assumption that the lengths of the intervals between successive crossings of the level M are uncorrelated.

Let us assume that one *knows* the two following quantities $A(t)$ and $N_<(t)$ from experiment, numerical simulations, or even analytically: $A(t)$ is the autocorrelation function of $\theta[M - X(t)]$ (θ is Heaviside's function),

$$A(t) = \langle \theta[M - X(t)]\theta[M - X(0)]\rangle, \tag{1}$$

and $N_<(t)$ is the average number of crossings at level M up

to time t, averaged over the starting position $X(0) < M$. For large time, one has

$$N_<(t) \sim N(t) = \frac{t}{\tau}, \tag{2}$$

with

$$N(t) = \left\langle \int_0^t |X'(t')| \delta(X(t') - M) dt' \right\rangle. \tag{3}$$

If the process is smooth, $X'(t')$ is not correlated with $X(t')$, and using stationarity, we find $N(t) = t \times \langle |X'(t)| \rangle \times \langle \delta[X(t) - M] \rangle$, which leads to

$$\tau^{-1} = g(M) \langle |X'(t)| \rangle. \tag{4}$$

In addition, $N_>(t)$, the average number of crossings at level M up to time t, starting from $X(0) > M$, satisfies the sum rule

$$G(M) N_<(t) + [1 - G(M)] N_>(t) = \frac{t}{\tau}, \tag{5}$$

where $G(M) = \int_{-\infty}^{M} g(x) dx$. Note that $\tau_\pm$ are simply related to $\tau = \frac{\tau_+ + \tau_-}{2}$:

$$\tau_- = 2\tau G(M), \qquad \tau_+ = 2\tau [1 - G(M)]. \tag{6}$$

In the following, we obtain closed forms for $P_<(t)$, $P_>(t)$, and $P_\pm(t)$ from the knowledge of $A(t)$ and $N_<(t)$, for any level M. When X is a Gaussian process of correlator $f(t)$, we shall see later that $A(t)$, $N_<(t)$, $N_>(t)$, τ, and $\tau_\pm$ can be explicitly written in terms of f. Hence, the minimal knowledge of the two-time correlation function of a Gaussian process will grant access to the yet unknown quantities $P_<(t)$, $P_>(t)$, and $P_\pm(t)$. However, the present approach has a wider range of applications and *does not rely* on the Gaussian property of the process.

Our central approximation consists in assuming that the *interval length between crossings are uncorrelated* [10]. The probability $P_<(N, t)$, starting from $X(0) < M$, that there are exactly N crossings in the interval $[0, t]$, can then be written, for odd $N = 2n - 1$ ($n \geq 1$),

$$P_<(2n-1, t) = \tau_-^{-1} \int_0^t dt_1 Q_-(t_1) \int_{t_1}^t dt_2 P_+(t_2 - t_1) \int_{t_2}^t dt_3 P_-(t_3 - t_2) \cdots \int_{t_{2n-3}}^t dt_{2n-2} P_+(t_{2n-2} - t_{2n-3}) \times \int_{t_{2n-2}}^t dt_{2n-1} P_-(t_{2n-1} - t_{2n-2}) Q_+(t - t_{2n-1}), \tag{7}$$

where $Q_\pm(t) = \int_t^{+\infty} P_\pm(t') dt'$ is the probability that a $\pm$ interval is larger than t. For even $N = 2n$ ($n \geq 1$), one obtains a similar expression

$$P_<(2n, t) = \tau_-^{-1} \int_0^t dt_1 Q_-(t_1) \int_{t_1}^t dt_2 P_+(t_2 - t_1) \int_{t_2}^t dt_3 P_-(t_3 - t_2) \cdots \int_{t_{2n-2}}^t dt_{2n-1} P_-(t_{2n-1} - t_{2n-2}) \times \int_{t_{2n-1}}^t dt_{2n} P_+(t_{2n} - t_{2n-1}) Q_-(t - t_{2n}). \tag{8}$$

For any function of time $F(t)$ introduced in this Letter, one defines its Laplace transform $\hat{F}(s) = \int_0^{+\infty} F(t) e^{-st} dt$. The convolution products in Eqs. (7) and (8) take a much simpler form in the Laplace variable s

$$\hat{P}_<(2n-1, s) = \tau_-^{-1} \hat{Q}_+ \hat{Q}_- [\hat{P}_+ \hat{P}_-]^{n-1}, \tag{9}$$

$$\hat{P}_<(2n, s) = \tau_-^{-1} \hat{Q}_-^2 P_+ [\hat{P}_+ \hat{P}_-]^{n-1}, \tag{10}$$

where $\hat{Q}_\pm(s) = \frac{1 - \hat{P}_\pm(s)}{s}$. One can now express the conservation of probability, $P_<(t) + \sum_{N=1}^{+\infty} P_<(N, t) = 1$, which leads to

$$\hat{P}_<(s) = \frac{1}{s} - \frac{1 - \hat{P}_-(s)}{\tau_- s^2}. \tag{11}$$

In fact, Eq. (11) is an *exact* relation, which reads

$$P_<(t) = \tau_-^{-1} \int_t^{+\infty} (t' - t) P_-(t') dt', \tag{12}$$

in the time variable. Indeed, if $X(t)$ has not crossed the level M up to time t, it belongs to a $-$ interval of duration $t' > t$, starting at an initial position uniformly distributed between 0 and $t' - t$. Note that $\hat{P}_>(s)$ and $\hat{P}_>(N, s)$ are given by similar expressions as Eqs. (9)–(11) by exchanging the indices $-$ and $+$.

Now, $\hat{P}_\pm(s)$ can be calculated by expressing the *known* quantities $\hat{A}(s)$ and $\hat{N}_<(s)$ as a function of $\hat{P}_\pm(s)$:

$$\hat{N}_<(s) = \frac{(1 + \hat{P}_+)(1 - \hat{P}_-)}{\tau_- s^2 (1 - \hat{P}_+ \hat{P}_-)}, \tag{13}$$

$$\hat{A}(s) = G(M) \left[\frac{1}{s} - \frac{1 - \hat{P}_+}{1 + \hat{P}_+} N_<(s) \right]. \tag{14}$$

Using $\hat{P}'_\pm(0) = -\tau_\pm$ and Eq. (6), one obtains the following estimates, valid for small s,

$$\hat{N}_<(s) \sim \frac{1}{\tau s^2}, \qquad \hat{A}(s) \sim \frac{G^2(M)}{s}. \tag{15}$$

The first expression in Eq. (15) is equivalent to Eq. (2), whereas the second relation expresses that for large t, $A(t) \sim G^2(M)$. For large s, $\hat{N}_<(s) \sim [2G(M)\tau s^2]^{-1}$, which corresponds to the small time behavior

$$N_<(t) \sim \frac{t}{2G(M)\tau}, \qquad N_>(t) \sim \frac{t}{2[1 - G(M)]\tau}. \tag{16}$$

For $G(M) \neq \frac{1}{2}$ [i.e., $M \neq 0$, when $g(X)$ is symmetric], Eq. (16) differs from the large time asymptotics given by

Eq. (2). However, the sum rule of Eq. (5) is preserved by the small time estimates of Eq. (16). Finally, writing

$$\hat{F}(s) = \frac{G(M) - sA(s)}{G(M)sN_<(s)}, \tag{17}$$

and using Eqs. (13) and (14), the interval distributions read

$$\hat{P}_+(s) = \frac{1 - \hat{F}(s)}{1 + \hat{F}(s)}, \tag{18}$$

$$\hat{P}_-(s) = \frac{2 - \tau_- s^2 N_<(s)[1 + \hat{F}(s)]}{2 - \tau_- s^2 N_<(s)[1 - \hat{F}(s)]}. \tag{19}$$

Inserting these expressions of $P_\pm$ in Eq. (11), one obtains our final result for $P_<$ (and $P_>$), from the sole knowledge of $A(t)$ and $N_<(t)$ [or $N_>(t)$].

The persistence exponent θ is defined as the asymptotic decay rate of $P_<(t) \sim e^{-\theta t}$. The term "exponent" arises from the fact that in many physical systems [7,8,10–17,20], the process X of interest is stationary in the variable $t = \ln T$, where T is the actual physical time. Thus the persistence decays as a power law $P_<(T) \sim T^{-\theta}$, as a function of the real time T (see below for practical examples). Within our approach, $-\theta$ is the first pole of $\hat{P}_<(s)$ [or equivalently of $\hat{P}_-(s)$] on the negative real axis [10]. Using Eq. (19), one finds that θ satisfies the implicit equation

$$\theta G(M)[1 + \theta N_<(-\theta)] + \theta^2 A(-\theta) = \tau^{-1}. \tag{20}$$

When M is large, θ goes to zero, and Eq. (20) leads to

$$\theta = \tau_-^{-1} = [2\tau G(M)]^{-1}. \tag{21}$$

For a Gaussian process, the same expression was obtained from a heuristic argument in [3]. In this limit of large M, the interval distributions are found to become Poissonian. We conjecture that the present approach becomes exact for large M, the $-$ intervals being so large that the $+$ intervals are indeed uncorrelated.

Let us move on to the case where $X(t)$ is a stationary *Gaussian* process. The properties of $X(t)$ [and hence $A(t)$ and $N_<(t)$] are completely determined by the sole knowledge of its two-time correlator $f(t) = \langle X(0)X(t)\rangle$. For a general process, this connection is only approximate and can only be made by means of the IIA [21]. For convenience, we set $\langle X^2(t)\rangle = f(0) = 1$. The process is smooth if f is twice differentiable. We also assume that for large time t, the correlator $f(t)$ decays fast enough so that $\int_0^\infty f(t)dt$ is finite [2,10]. For $t > 0$, the position-velocity correlator is $\langle X(0)X'(t)\rangle = f'(t)$, which vanishes for $t = 0$, since $f(t)$ is an even function, twice differentiable at $t = 0$. The velocity-velocity correlation function is $\langle X'(0)X'(t)\rangle = -f''(t)$. The mean time interval between crossings τ is computed using Eq. (4)

$$\tau^{-1} = \frac{\sqrt{-f''(0)}}{\pi} e^{-M^2/2}. \tag{22}$$

For a Gaussian process, $A(t)$ has been derived in [20]

$$A(t) = \int_{-\infty}^{M} g(x) G\left(\frac{M - xf(t)}{\sqrt{1 - f^2(t)}}\right) dx. \tag{23}$$

For large time, so that $f(t)$ is small, one finds

$$A(t) = G^2(M) + \frac{f(t)}{2\pi} e^{-M^2} + O[f^2(t)]. \tag{24}$$

Finally, $N_<(t)$ can be calculated after introducing the correlation matrix of the Gaussian vector $(X(t), X(0), X'(t))$, which reads

$$\mathcal{C}(t) = \begin{pmatrix} 1 & f(t) & 0 \\ f(t) & 1 & f'(t) \\ 0 & f'(t) & -f''(0) \end{pmatrix}. \tag{25}$$

One finds

$$N_<(t) = G^{-1}(M) \int_0^t \langle |X'(t')| \rangle_< dt', \tag{26}$$

where $\langle |X'(t)| \rangle_<$ is the average of the velocity modulus, knowing that $X(t) = M$, and averaged over $X(0) < M$:

$$\langle |X'(t)| \rangle_< = \int_{-\infty}^{M} dx_0 \int_{-\infty}^{+\infty} dv \frac{|v| e^{-(1/2)\mathbf{U}^\dagger \mathcal{C}^{-1} \mathbf{U}}}{(2\pi)^{3/2} \sqrt{\det \mathcal{C}}}, \tag{27}$$

where $\mathbf{U} = (M, x_0, v)$. In practice, this integral has to be computed numerically, but can be reduced to a cumbersome one-dimensional integral over x_0, involving g and G. For large time or small $f(t)$, one obtains

$$\langle |X'(t)| \rangle_< - \frac{1}{\tau} = -\frac{Mg(M)}{G(M)\tau} f(t) + O(f^2(t)). \tag{28}$$

Note that for numerical purposes, the Laplace transform of $N_<(t)$ can be efficiently written as

$$\hat{N}_<(s) = \frac{1}{s} \int_0^{+\infty} \left(\langle |X'(t)| \rangle_< - \frac{1}{\tau} \right) e^{-st} dt + \frac{1}{\tau s^2}. \tag{29}$$

The various analytical asymptotic forms obtained above can be useful to complement the partial knowledge of $A(t)$ and $N_<(t)$ from a partial experimental or numerical sampling of the process $X(t)$ [21].

As an application, Table I reports the theoretical and numerical values (the latter obtained by direct simulation of the temporal process) of the persistence exponent θ for

TABLE I. Exponents θ from theory (θ^{th}) and simulations (θ^{sim}), for different values of M, calculated for the processes X_1 and X_2 introduced in the text. For $M = 0$, the three first results were first reported in [10], while the exact value $\theta_2(M = 0) = \frac{1}{4}$ was obtained in [22].

M	θ_1^{th}	θ_1^{sim}	θ_2^{th}	θ_2^{sim}
0	0.1862	0.188(1)	0.2647	1/4
1	5.914×10^{-2}	$5.91(1) \times 10^{-2}$	8.625×10^{-2}	$8.31(4) \times 10^{-2}$
2	1.084×10^{-2}	$1.09(1) \times 10^{-2}$	1.665×10^{-2}	$1.61(2) \times 10^{-2}$
3	8.769×10^{-4}	$8.77(2) \times 10^{-4}$	1.420×10^{-3}	$1.42(1) \times 10^{-3}$

$M = 0, 1, 2, 3$ and for the two Gaussian processes X_1 and X_2 associated to the correlators

$$f_1(t) = \frac{1}{\cosh(\frac{t}{2})}, \tag{30}$$

$$f_2(t) = \tfrac{1}{2}(3e^{-|t|/2} - e^{-3|t|/2}). \tag{31}$$

X_1 and X_2 are two examples of non-Markovian Gaussian processes arising from physical systems. Indeed, up to a multiplicative term, $X_1(t)$ can be shown to be equal to $\rho(\mathbf{x}, T)$ (at some arbitrary position $\mathbf{x}$), where the density field $\rho(\mathbf{x}, T)$ evolves according to the two-dimensional diffusion or heat equation [10], $\frac{\partial \rho}{\partial T} = \nabla^2 \rho$, starting from an arbitrary (although not too strongly correlated) initial condition. Here, the actual time T is again related to our stationary time t by the relation $t = \ln T$. Note that the out of equilibrium dynamics of a two-dimensional Ising model after a quench at a temperature $T_0 < T_c$ (where T_c is the ferromagnetic critical temperature) can be approximately mapped to this problem [10,20], by assimilating the spin $S(\mathbf{x}, T) = \pm 1$ to $S(\mathbf{x}, T) = \mathrm{sgn}[\rho(\mathbf{x}, T)]$. We find that the theoretical values are within the numerical error bars, except maybe for $M = 0$. Overall, the accuracy is better than 1%. In addition, the asymptotic result of Eq. (21) already leads to fair estimates for $M = 2$ and $M = 3$ ($\theta_1(M = 2) = 1.102 \times 10^{-2}$ and $\theta_1(M = 3) = 8.852 \times 10^{-4}$). Our theoretical and numerical results are also consistent with the numerical bounds computed in [3], for $M = 1$ and $M = 2$ [$0.0586 < \theta_1(M = 1) < 0.0684$ and $0.0106 < \theta_1(M = 2) < 0.0119$].

As for X_2, it is associated with the random acceleration process [22], $\frac{d^2X}{dT^2} = \eta(T)$, where $\eta(T)$ is a δ-correlated white noise, in the variable $T = e^t$. Contrary to X_1, the process X_2 is not infinitely differentiable [$f_2'''(t)$ is not defined at $t = 0$], although it is just smooth enough for the present approach to be applicable. Hence, it is not surprising that the theoretical results are not as good as for the process X_1. However, the theoretical estimates are clearly becoming more accurate as M increases, and presumably exact for large M.

In conclusion, the present work develops a powerful approximation leading to explicit expressions for the Laplace transform of the probability to remain above or below a certain level M (and hence the distribution of the minimum or maximum of the process). This approach also gives the distribution of time intervals during which the process remains above or below M, and leads to the determination of the persistence exponent.

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