Unifying Quantum State Transfer and State Amplification

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We present a Hamiltonian that can be used for amplifying the signal from a quantum state, enabling the measurement of a macroscopic observable to determine the state of a single spin. We prove a general mapping between this Hamiltonian and an exchange Hamiltonian for arbitrary coupling strengths and local magnetic fields. This facilitates the use of existing schemes for perfect state transfer to give perfect amplification. We further prove a link between the evolution of this fixed Hamiltonian and classical cellular automata, thereby unifying previous approaches to this amplification task. Finally, we show how to use the new Hamiltonian for perfect state transfer in the scenario where total spin is not conserved during the evolution, and demonstrate that this yields a significantly different response in the presence of decoherence.

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Introduction.—One of the many challenging tasks in realizing technology for quantum information processing is measuring the output of a protocol. Typically, the result is expected to be stored on single spins, when even detecting the presence of a single spin is an experimental challenge, let alone measuring its internal state. Recently, there have been proposals for amplifying a quantum state so that it is converted into a macroscopic property which can be measured [1–3]. Since copying the state many times is impossible, the aim of these protocols is to perform the conversion

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle^{\otimes(N-1)} \rightarrow \alpha|0\rangle^{\otimes N} + \beta|1\rangle^{\otimes N}. \quad (1)$$

If the states $|0\rangle$ and $|1\rangle$ are stored on magnetic sublevels of the spins, this yields a macroscopic magnetization which can be measured to determine if the original state was $|0\rangle$ or $|1\rangle$. The original proposal focused on using a fixed Hamiltonian that achieved some macroscopic change, although did not manage the desired transformation with unit fidelity. Subsequently, the idea of using techniques from classical cellular automata (CA) was proposed [3], which realized a speedup by using a cubic organization of spins instead of a linear geometry.

In this Letter, we provide a unification of these techniques in a one-dimensional system. Making use of existing work on perfect state transfer in spin chains, primarily from [4–7], a large set of results can be rephrased in the context of this system. This leads to the development of a fixed Hamiltonian, which not only achieves the evolution in Eq. (1), but accurately reconstructs the results of the cellular automaton for all possible initial states of the system. This new Hamiltonian can, itself, be used for perfect state transfer without any external interaction, and is the first example that does not preserve the total spin during the process. As such, its behavior can be expected to be significantly different when, for example, noise is present.

Signal amplification.—In the one-dimensional case, the CA approach worked by applying a series of commands to the chain of spins. These commands were capable of detecting a local sequence of either $|1x0\rangle$ or $|0x1\rangle$ and converting them into $|1\bar{x}0\rangle$ and $|0\bar{x}1\rangle$, respectively. Ensuring that $x = 0$ guarantees amplification (an increase in the number of 1's, providing there was already at least one 1 present). This was achieved by alternating the application of these pulses to even and odd qubits on the chain. There is a corresponding Hamiltonian which can achieve the same result;

$$K_n = \tfrac{1}{2}(X_n - Z_{n-1}X_nZ_{n+1}),$$

where X_n is the Pauli X rotation applied to qubit n. The CA commands that are applied at either end of the chain are slightly different. Since the first qubit should never be flipped, there is no initial term. At the end of the chain, we want to convert from $|10\rangle$ to $|11\rangle$; hence, we set

$$K_N = \tfrac{1}{2}(\mathbb{1} - Z)_{N-1}X_N.$$

The CA proceeds by alternately applying $\sum_n K_{2n}$ and $\sum_n K_{2n+1}$. Instead, we wish to create a fixed Hamiltonian that does not require this alternation of terms. As noted in [1], the Hamiltonian

$$H = \sum_{n=2}^{N} J_{n-1}K_n$$

keeps the state $|100\ldots0\rangle$ in a subspace which we describe as

$$|\tilde{n}\rangle := \Big|\overbrace{1\ldots1}^{n}\overbrace{0\ldots0}^{N-n}\Big\rangle. \quad (2)$$

Calculating the action of the Hamiltonian, $H|\tilde{n}\rangle = J_{n-1}|\widetilde{n-1}\rangle + J_n|\widetilde{n+1}\rangle$, we observe that this is identical to the action of an exchange Hamiltonian

$$H_{\mathrm{ex}} = \frac{1}{2}\sum_{n=1}^{N-1} J_n(X_nX_{n+1} + Y_nY_{n+1}),$$

on a single excitation located on site n, $H_{\mathrm{ex}}|n\rangle = J_{n-1}|n-1\rangle + J_n|n+1\rangle$. Hence, if we were to set $J_n = 1$, we would immediately recover the results of [4,7] for state transfer using a uniformly coupled chain ($|1\rangle \rightarrow |N\rangle$), but in the case of signal amplification. In particular, we recover the same calculations as [1], and the fact that the goal of Eq. (1) can never be perfectly realized in chains of more than 3 spins with uniform couplings. We also discover how to perfectly realize the process $|\tilde{1}\rangle \rightarrow |\tilde{N}\rangle$ in a time $t_0 = \pi/2$, by engineering the couplings $J_n = \sqrt{n(N-n)}$ [5,7]. The state $|0\rangle$ remains unchanged. Recent results show that this choice of couplings optimizes the speed of transfer when applying constraints, such as limiting the maximum energy of the Hamiltonian [8], or the maximum coupling strength [9]. Applying either of these yields the scaling relation $t_0 \sim N$.

The arrival probability of the initial state $|\tilde{1}\rangle$ in the final state $|\tilde{N}\rangle$ is given by

$$\sin^{2(N-1)}\left(\frac{\pi t}{2t_0}\right).$$

As the number of spins increases, keeping t_0 constant, the sharpness of the arrival peak is enhanced, requiring increasingly accurate timing for the measurement. This is not a drawback for the scheme, however, since we can also calculate the average signal strength,

$$\frac{1}{N} + \frac{N-1}{N}\sin^2\left(\frac{\pi t}{2t_0}\right).$$

This depends only weakly on N near the peak, so we expect that the protocol is largely insensitive to timing errors. This is further justified by the calculation of the probability that more than half of the spins have been flipped [10], as depicted in Fig. 1. From this, we conclude that provided we only need $N/2$ spins to be flipped to detect the signal, there is a window of t_0 in which the measurement can be performed. Further robustness is present, as it was in [2], since, as we wish to perform a measurement at the end, coherence between the $|0\rangle$ and $|1\rangle$ components need not be preserved.

Many other results from the study of state transfer can readily be applied to obtain perfect, or near-perfect, amplification under different constraints, depending on the experimentally available control. For example, a local magnetic field [11] could be used to enhance the amplification on the uniformly coupled Hamiltonian [1], where we replace the local magnetic fields $\sum B_nZ_n$ of the exchange model with either $\sum B_nZ_nZ_{n+1}$ terms [12] or local magnetic fields $\sum B'_nZ_n$ such that $B'_n = B_{n-1} - B_n$. Alternatively, we could tune the couplings J_n and magnetic fields B_n to yield different spectra [8,13], which enable control of a range of useful properties such as the robustness against a variety of errors. Naturally, schemes that require single spin measurement to herald the correct evolution, such as [14,15], should not be used.

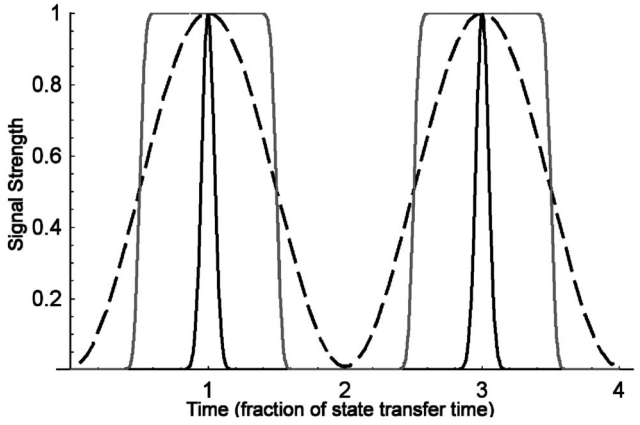

FIG. 1. Signal amplification for a system of $N = 100$ qubits. Perfect amplification occurs at times $(2n+1)t_0$ and revival of the initial state occurs at times $2nt_0$. The solid line indicates the probability of the signal (initially $|\tilde{1}\rangle$) being amplified to the desired final state $|\tilde{N}\rangle$. The dashed line shows the average signal strength (as a fraction of the maximum strength), and the gray line plots the probability that more than half of the spins have been flipped. Both the dashed and gray lines are largely independent of the number of qubits in the system.

Cellular automata.—Having demonstrated the equivalence of the Hamiltonians H and H_{ex} in the 0th and 1st excitation subspaces, we shall proceed to examine the general equivalence. The basis states of the second excitation subspace of H_{ex} are denoted by $|n, m\rangle$, describing excitations at the sites n and m. The action of H on the state

$$|\tilde{n}, \tilde{m}\rangle := |\tilde{n} \oplus \tilde{m}\rangle$$

is identical (for example, $|\tilde{3}, \tilde{5}\rangle = |11\,100 \oplus 11\,111\rangle = |00\,011\rangle$). The generalization to higher excitation subspaces is straightforward, constituting bitwise addition modulo 2 of the effective single excitations. Formally, the equivalence of the two Hamiltonians is proven by demonstrating a transformation between them [16]. This transformation is constructed by repeatedly applying controlled-NOT gates C_n^{n-1} to H_{ex} with control qubit n and target qubit $n-1$ starting from $n = N$ and finishing with $n = 2$,

$$C_2^1 C_3^2 \ldots C_N^{N-1} H_{\mathrm{ex}} C_N^{N-1} \ldots C_2^1 = H.$$

The equality with H follows from the standard propagation properties of the Pauli matrices through the controlled-NOT gate [17], where Z propagates from target to control and X from control to target. Hence, the X_nX_{n+1} terms become X_{n+1} and the Y_nY_{n+1} terms become $-Z_nX_{n+1}Z_{n+2}$, except for the final term, which transforms into $-Z_{N-1}X_N$, thereby recovering H. The same transformation can also be used to show how local magnetic fields transform, and subsequently allows us to describe the subspace structure

of H,

$$\left[H, \sum_{n=1}^{N-1} Z_n Z_{n+1} + Z_N\right] = 0.$$

The conserved quantity is no longer the number of excitations in the system, but the number of domain boundaries.

As a consequence of the equivalence of H and H_{ex}, not only can we use results for the single excitation subspace of state transfer chains, but for all excitation subspaces [6]. In particular, for the engineered couplings $J_n = \sqrt{n(N-n)}$, any classical initial state of the chain (not just $|0\rangle^{\otimes N}$ and $|1\rangle|0\rangle^{\otimes(N-1)}$) yields a classical output, because perfect state transfer occurs in all excitation subspaces. This output is precisely that given by the cellular automaton. The action of a CA command is

$$|x, y, z\rangle \rightarrow |x, x \oplus y \oplus z, z\rangle$$

on every second qubit. By construction, the action of the Hamiltonian on the effective single excitation subspace corresponds to the CA. Therefore, and as a consequence of the facts that bitwise addition operation is commutative, and our basis states are correctly described by bitwise addition, the output must be the same as the CA for the whole space of states.

The Hamiltonian H is, up to the local terms $\frac{1}{2} \times \sum_{n=2}^{N} J_{n-1} X_n$, the cluster state Hamiltonian, i.e., the Hamiltonian that has the cluster state as its ground state. It has been proven [18] that 3-body terms are a necessity for any such Hamiltonian. This must be true, not only for the ground state, but also for the excited states because the excited states can be turned into ground states by changing the signs of the constants J_n. Consequently, we can conclude that any Hamiltonian which is to give the same evolution as H must consist of at least 3-body terms. This is true whatever the coupling strengths J_n because $[K_n - \frac{1}{2} X_n, K_m - \frac{1}{2} X_m] = 0$ and hence the coupling strengths only determine the spectrum of the cluster state Hamiltonian, not the eigenstates themselves. Nevertheless, it is possible to construct 2-body Hamiltonians that have a cluster state as the ground state of logical qubits [19]. Were a crystal with a suitable natural Hamiltonian to be found, this would directly implement uniform coupling schemes that achieve approximate evolution. Perfect, near-uniform coupling schemes [13] might then be realized, since they only require small local fluctuations in field strength, which could perhaps be introduced by impurities.

Perfect state transfer.—With the definition of higher excitations as stated, we can readily see that our Hamiltonian also performs perfect state transfer. A single excitation at site $n \neq 1$ is in the state $|\widetilde{n-1}, \tilde{n}\rangle$, and transfers to $|N + \widetilde{1} - n, N + \widetilde{2} - n\rangle$. The effect of fermion exchange [6,20] now manifests itself as a Z gate on the state. Furthermore, we can choose to transfer several states at once, and while perfect transfer still occurs, there is no controlled-phase gate between the exchanging states. While potentially useful for some purposes [7,21], this exchange phase is undesirable in the case of state transfer, so this Hamiltonian provides an alternative to encoding the qubits in pairs of spins $|0_L\rangle = |01\rangle$ and $|1_L\rangle = |10\rangle$ on the same chain, as is required for all previous chains. This behavior, which can be viewed as a consequence of the fact that the Hamiltonian is not spin preserving, can be expected to affect other properties of the chain. In particular, one might expect it to have significantly different resistance to decoherence than previous chains. In Fig. 2, we examine dephasing noise during state transfer using the two different Hamiltonians and observe that the new Hamiltonian exhibits a significant enhancement in robustness in this special case.

By making certain assumptions relating to, for example, the maximum coupling strength, the amplification time $t_0 \sim N$. In an attempt to improve this, one could examine the case of higher-dimensional geometries, as one might expect a d-dimensional cube to have a transfer time $t_0 \sim \sqrt[d]{N}$ [3]. However, this fails for two principal reasons. First, there are no known geometries other than a simple chain that perfectly transfer states in all excitation subspaces at the same time [10]. Indeed, it is possible to prove that perfect state transfer is impossible for a large class of such cases [22]. Second, the chain splitting techniques demonstrated in [10,23,24] do not work with the new Hamiltonian if we restrict to three-body terms. This splitting technique functioned by observing that if two spins are each coupled to a single spin with strength $J/\sqrt{2}$, then the state $|01\rangle + |10\rangle$ across this pair can be treated by replacing the pair of qubits with a single qubit, coupled with strength J, and a single excitation. Using this technique in the present setting, we still get a superposition of $|01\rangle + |10\rangle$, and not the

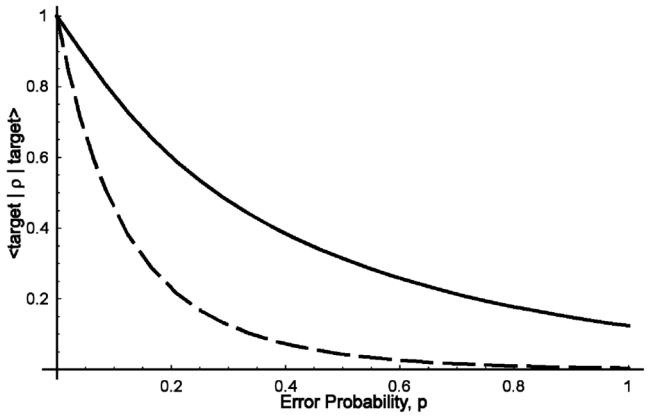

FIG. 2. Comparison of state transfer across 6 spins using H (solid line) and H_{ex} (dashed line) in the presence of dephasing noise. We have assumed that a single phase flip occurs with the probability p at each of 25 time steps on a random qubit. The initial state is a single excitation, and the measured fidelity is the probability that an excitation is found on the expected output qubit.

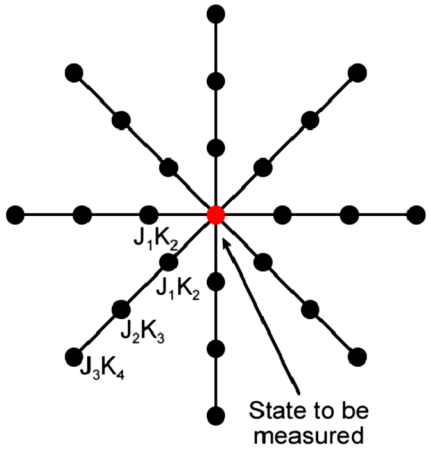

FIG. 3 (color online). Star-shaped geometry which consists of the spin to be measured in the center, and a set of one-dimensional chains radiating outwards. This achieves perfect amplification because the only terms that are applied to the central spin are Z, which commute. Note that the coupling strengths do not need to be reduced, as in [10].

$|11\rangle$ which would be required for amplification. Some improvement can be gained by considering the geometry illustrated in Fig. 3, where the signal gets enhanced by a factor R if there are R spikes on the star. The different spikes do not compete with each other because the Hamiltonians H only interact on the qubit where the state is initially stored, and on this qubit, they commute because all terms are Z and $\mathbb{1}$. Alternatively, we could say that to get a particular signal size, we require N qubits, and hence the time required for the protocol is reduced to $t_0 \sim N/R$.

Summary and further work.—In summary, we have presented a map between an exchange Hamiltonian and a clusterlike Hamiltonian which enables previous ideas on perfect state transfer to be directly applied to the problem of signal amplification. It has also yielded a new method for state transfer, which is not restricted to being spin preserving, and avoids the problems of fermionic exchange when multiple states are transferred. Intriguingly, the presented Hamiltonian implements a discrete CA in a continuous-time system.

The present work raises several interesting questions. For example, is it possible to find Hamiltonians that correctly simulate quantum cellular automata (QCA) which issue commands such as "apply the operation U if your neighbors are different", perhaps by using a new Hamiltonian of the form

$$H_{\mathrm{QCA}} = \sum_n J_{n-1}(H_n - Z_{n-1}H_nZ_{n+1}),$$

where the coupling strengths J_n and local Hamiltonians H_n would need to be determined. The existing work at least justifies that there will be a similar subspace restriction. On a more wide-reaching basis, we have succeeded in demonstrating that state transfer ideas need not be restricted to spin preserving Hamiltonians, as previously thought. Consequently, are there any other useful protocols to which we can apply similar ideas? One potential candidate is the optimal universal cloning machine. Previous attempts, when restricted to spin preserving Hamiltonians, have only succeeded in creating phase-covariant cloners [25].

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[1] J.-S. Lee and A. K. Khitrin, Phys. Rev. A **71**, 062338 (2005).
[2] P. Cappellaro, J. Emerson, N. Boulant, C. Ramanathan, S. Lloyd, and D. G. Cory, Phys. Rev. Lett. **94**, 020502 (2005).
[3] C. A. Perez-Delgado, M. Mosca, P. Cappellaro, and D. G. Cory, Phys. Rev. Lett. **97**, 100501 (2006).
[4] S. Bose, Phys. Rev. Lett. **91**, 207901 (2003).
[5] M. Christandl, N. Datta, A. Ekert, and A. J. Landahl, Phys. Rev. Lett. **92**, 187902 (2004).
[6] C. Albanese, M. Christandl, N. Datta, and A. Ekert, Phys. Rev. Lett. **93**, 230502 (2004).
[7] M. Christandl, N. Datta, T. Dorlas, A. Ekert, A. Kay, and A. Landahl, Phys. Rev. A **71**, 032312 (2005).
[8] A. Kay, Phys. Rev. A **73**, 032306 (2006).
[9] M.-H. Yung, Phys. Rev. A **74**, 030303(R) (2006).
[10] A. Kay and M. Ericsson, New J. Phys. **7**, 143 (2005).
[11] T. Shi, Y. Li, Z. Song, and C. P. Sun, Phys. Rev. A **71**, 032309 (2005).
[12] This is the strict equivalence that holds for all excitation subspaces.
[13] P. Karbach and J. Stolze, Phys. Rev. A **72**, 030301(R) (2005).
[14] D. Burgarth and S. Bose, Phys. Rev. A **71**, 052315 (2005).
[15] D. Burgarth and S. Bose, New J. Phys. **7**, 135 (2005).
[16] We thank Michael Nielsen for suggesting this method.
[17] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000), 5th ed.
[18] M. A. Nielsen, quant-ph/0504097.
[19] S. D. Bartlett and T. Rudolph, Phys. Rev. A **74**, 040302(R) (2006).
[20] M.-H. Yung and S. Bose, Phys. Rev. A **71**, 032310 (2005).
[21] S. Clark, C. Moura-Alves, and D. Jaksch, New J. Phys. **7**, 124 (2005).
[22] A. Kay (to be published).
[23] A. Perales and M. B. Plenio, J. Opt. B **7**, S601 (2005).
[24] S. Yang, Z. Song, and C. P. Sun, Eur. Phys. J. B **52**, 377 (2006).
[25] G. De Chiara, R. Fazio, C. Macchiavello, S. Montanegro, and G. M. Palma, Phys. Rev. A **70**, 062308 (2004).