Nonlinear Coupling of Nanomechanical Resonators to Josephson Quantum Circuits

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We propose a technique to couple the position operator of a nanomechanical resonator to a SQUID device by modulating its magnetic flux bias. By tuning the magnetic field properly, either linear or quadratic couplings can be realized, with a discretely adjustable coupling strength. This provides a way to realize coherent nonlinear effects in a nanomechanical resonator by coupling it to a Josephson quantum circuit. As an example, we show how squeezing of the nanomechanical resonator state can be realized with this technique. We also propose a simple method to measure the uncertainty in the position of the nanomechanical resonator without quantum state tomography.

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Introduction.—Historically, mechanical systems have not been the favorite proving ground of quantum mechanics because of their macroscopic nature. This situation has changed in recent years thanks to the impressive advance of nanofabrication technologies. It is now possible to make nanomechanical resonators with frequencies of gigahertz [1] and quality factors approaching 10^5 at millidegree Kelvin temperatures [2]. This opens the possibility of studying coherent quantum behavior in mesoscopic mechanical systems [3,4]. One can also exploit the quantum properties of mechanical degrees of freedom for applications in areas such as weak force detection [5], precision measurement [6], and quantum information processing [7]. Recently, evidence for quantized displacement in a nanomechanical resonator has been observed [8].

All micro and nanomechanical devices require some means of transduction. As discussed in Ref. [3], an excellent way to engineer and detect the quantum modes of a nanomechanical resonator is to couple the resonator to Josephson-device–based solid state circuitry [9] on which quantum coherent control has been demonstrated. A straightforward coupling scheme for this purpose is to voltage bias the resonator and use the position dependent electrostatic interaction between the nanoresonator and a charge island. This is discussed in Ref. [3] and used in all previous studies. Though conceptually simple and practically feasible, this scheme has the limitation that the dominant term in the induced coupling is always linear in the resonator position. Consequently, it can only be used to realize linear effects in the nanoresonator. As is well-known in quantum optics [10], nonlinear effects are indispensable for thorough study and control of the dynamics of harmonic oscillators. They are required to produce essential processes such as squeezing and parametric amplification. They are also known to be necessary for universal quantum information processing on continuous variables [11]. Thus, it is highly desirable to develop new coupling schemes which can introduce nonlinear effects into the nanoresonator system.

In this work, we propose an attractive alternative method to couple the nanomechanical resonator to a Josephson quantum circuit by modulating the flux bias of a SQUID device. Our scheme makes it possible to realize both linear and nonlinear processes on the nanomechanical resonator. We show how we can generate squeezing of the nanomechanical resonator using this method and propose a simple way to measure the reduction in the uncertainty in the resonator's position.

The device and its working principle.—Our scheme is illustrated in Fig. 1(a). Here, the nanomechanical resonator is in one arm of a SQUID device. The SQUID loop is biased with a perpendicular magnetic field B.

We see that the area of the SQUID loop is dependent on the position of the nanomechanical resonator. As the resonator oscillates, the flux bias of the SQUID changes leading to voltage variations across the SQUID. In Fig. 1(a), we denote the width and length of the SQUID loop W and L. The length and small displacement of the nanomechanical resonator are l and X, defined such that the area of the SQUID loop is $WL + lX$. The total flux bias of the SQUID loop is $\Phi_e^0 + BlX$, where $\Phi_e^0 = BWL$ is the flux bias corresponding to the equilibrium position of the nanomechanical resonator. With the phase drops of the two junctions being ϕ_1 and ϕ_2, the Josephson energy of the SQUID is $E_J = -E_J^0 \cos\phi_1 - E_J^0 \cos\phi_2 = -2E_J^0 \cos[(\phi_1 - \phi_2)/2] \cos[(\phi_1 + \phi_2)/2]$, where $E_J^0 = I_c \Phi_0 / 2\pi$ is the Josephson energy of the (identical) junctions, I_c the junction critical current, and Φ_0 the flux quantum. Since the

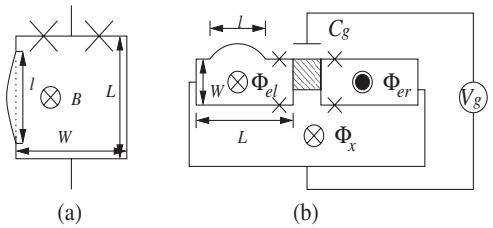

FIG. 1. (a) A nanomechanical resonator in one arm of a SQUID device threaded by a magnetic field. (b) A more sophisticated design in which a nanomechanical resonator is coupled to a charge qubit. The shaded box is the superconducting island of the charge qubit.

superconducting order parameter is single valued, we must have $\phi_1 - \phi_2 + 2\pi(\Phi_e^0 + BlX)/\Phi_0 = 2\pi p$ for some integer p leading to [12]

$$E_J = -2E_J^0 \cos\left(\frac{\pi\Phi_e^0}{\Phi_0} + \frac{\pi Bl}{\Phi_0}X\right)\cos\phi, \qquad (1)$$

where $\phi = (\phi_1 + \phi_2)/2$ is the average phase across the junctions.

It is clear from Eq. (1) that, in general, the Josephson energy of the SQUID is a nonlinear function of X, the position of the nanomechanical resonator. Treating X as a small variable, we find that there are two flux bias points of particular interest. When $\Phi_e^0 = (n + 1/2)\Phi_0$, n an integer, to lowest order in X the Josephson energy of the SQUID is $E_J = (-1)^n 2E_J^0(\pi(n + 1/2)l/WL)X\cos\phi$. In this case, E_J has a linear dependence on X. On the other hand, when $\Phi_e^0 = n\Phi_0$, to lowest order in X the Josephson energy is

$$E_J = -(-1)^n 2E_J^0\{1 - (\pi nl/WL)^2X^2/2\}\cos\phi. \qquad (2)$$

Here, the Josephson energy of the SQUID has a quadratic dependence on the nanoresonator position.

Our coupling scheme can be considered a magnetic transduction method. Compared to the electrostatic transduction scheme [3], the distinctive advantage of our scheme is that the coupling can be either linear or quadratic, depending on the flux bias point of the SQUID. Also, notice that the coupling strength can be adjusted (discretely). It is then possible, in principle, to operate in both linear and nonlinear, as well as both weak and strong, coupling regimes. In practice, the SQUID will be part of a Josephson quantum circuit, and the modulation scheme above then couples the nanomechanical oscillator to the Josephson quantum circuit.

Note the nanomechanical resonator must be superconducting. This can be realized by using a metalized resonator [2] as long as the magnetic field does not exceed the critical field strength of the superconductor.

Squeezing of the nanomechanical resonator.—An important nonlinear effect on a harmonic oscillator is squeezing. Our nonlinear coupling scheme makes it possible to realize squeezing and other coherent nonlinear processes on nanoresonators which are inaccessible by previous methods. This provides a way of introducing nonlinear-effect–induced operators, for instance, as gates within a complicated quantum circuit. Previously, generation of certain squeezed states of nanomechanical resonators has been studied by a few authors [13,14]. However, these schemes function incoherently and use dissipation and measurement to generate the needed nonlinearity. They are incapable of effecting unitary evolutions.

In the following we show how we can realize squeezing on a nanomechanical resonator using our scheme to couple it to a charge qubit [9]. For this purpose, we consider the design shown in Fig. 1(b). Here, we have two identical SQUIDs (left and right) biased at equal but opposite fluxes $\Phi_{el} = -\Phi_{er} = BWL$ [15]. The nanomechanical resonator is in the arm of one of the SQUIDs. The big loop is biased with an external flux Φ_x.

In order to realize nonlinear coupling between the nanomechanical resonator and the charge qubit, we bias the SQUIDs at $\Phi_{el} = -\Phi_{er} = n\Phi_0$. Using the same argument that leads to Eq. (2), we can derive the Josephson energy of the circuit in Fig. 1(b). To lowest order in the resonator position X, $E_J = -(-1)^n 4E_J^0\cos(\pi\Phi_x/\Phi_0)\cos\phi + (-1)^n E_J^0(\pi nl/WL)^2X^2\cos\phi_l$, where ϕ_l and ϕ_r are the phases of the left and right SQUIDs (the averages of the phases of the two junctions in the SQUIDs), and $\phi = (\phi_l + \phi_r)/2$ is the average phase of the SQUIDs conjugate to the charge number on the island. If we bias the big loop at $\Phi_x = (2m + 1/2)\Phi_0$, m an integer, only the coupling term survives, so $E_J = (-1)^n E_J^0(\pi nl/WL)^2X^2\cos\phi_l$.

The charge island possesses an adjustable gate voltage V_g which is applied through the gate capacitance C_g. When it is biased close to $n_g = C_gV_g/2e = 1/2$, the states with 0 and 1 excess Cooper pairs comprise the low energy Hilbert space of the qubit. Considering these charge states the spin up and spin down states [3] in an effective two state system, we can use the Pauli matrices to describe operators acting on the system. Its uncoupled Hamiltonian is $E_z\sigma_z/2$, where $E_z = (2n_g - 1)(2e)^2/2C_t$ and C_t is the total capacitance of the charge island.

As usual, the nanoresonator is treated as a harmonic oscillator with position operator $X = \delta X_0(a + a^\dagger)$ [3], where a is the annihilation operator, $\delta X_0 = \sqrt{\hbar/2M\omega_0}$ is the zero point fluctuation in the resonator's position X, and M and ω_0 are the mass and frequency of the resonator. Its uncoupled Hamiltonian is $\omega_0 a^\dagger a$.

The system Hamiltonian is then $H = E_z\sigma_z/2 + \omega_0 a^\dagger a - (\lambda_n/2)(a + a^\dagger)^2\sigma_y$, where the coupling strength $\lambda_n = -(-1)^n E_J^0(\pi nl\delta X_0/WL)^2$. If we choose $E_z = 2\omega_0$ and shift to the rotating frame defined by $E_z\sigma_z/2 + \omega_0 a^\dagger a$, we obtain the following Hamiltonian [10]:

$$H_R = i(\lambda_n/2)(a^2\sigma^+ - a^{\dagger 2}\sigma^-) + H_{\omega_0}, \qquad (3)$$

where $\sigma^\pm = (\sigma_x \pm i\sigma_y)/2$ and H_{ω_0} are off resonance terms of magnitude $\lambda_n/2$ oscillating at frequencies of $2\omega_0$ and higher. Since the realizable coupling strength λ_n is usually much smaller than the resonator frequency ω_0, we can adopt the rotating wave approximation to drop H_{ω_0}.

In addition to the operating point discussed above, we also consider another set of bias conditions in which the SQUIDs are biased at 0 flux and E_z is changed (by tuning the gate voltage V_g) to $E_z' = 2\omega_0 + \delta E_z$. We also bias the big loop slightly away from $(2m + 1/2)\Phi_0$ using a small ac field, $\Phi_x = (2m + 1/2)\Phi_0 + \delta\Phi_x\cos 2\omega_0 t$ where $\pi\delta\Phi_x/\Phi_0 \ll 1$. In this case the charge qubit is decoupled from the resonator. To lowest order in $\pi\delta\Phi_x/\Phi_0$, the system Hamiltonian in the same rotating frame is $H = \delta E_z\sigma_z/2 + E_x\sigma_x/2 + H'_{\omega_0}$, where $E_x = 4E_J^0(\pi\delta\Phi_x/\Phi_0)$

and the rapidly oscillating term H'_{ω_0} will have negligible effect if E_x is small compared to $4\omega_0$ and the system evolves for an appropriate duration [17]. Note both δE_z and E_x of the qubit can be adjusted [3]; therefore we can perform arbitrary rotations on the state of the charge qubit. In particular, if we choose large values for δE_z and E_x, we can realize pulsed operations on the charge qubit and flip its state quickly.

We now consider a spin-echo-like process in which we let the system evolve under the Hamiltonian (3) for short periods of time of duration Δt. In between each such time interval we apply a quick π pulse σ_x to the charge qubit to flip its state, so that the evolution for two periods is governed by $\exp\{-iH_R\Delta t\}\sigma_x \exp\{-iH_R\Delta t\}\sigma_x$. Since $\sigma_x \exp\{(\lambda_n\Delta t/4)[(a^2 - a^{\dagger 2})\sigma_x + i(a^2 + a^{\dagger 2})\sigma_y]\}\sigma_x = \exp\{(\lambda_n\Delta t/4)[(a^2 - a^{\dagger 2})\sigma_x - i(a^2 + a^{\dagger 2})\sigma_y]\}$, this evolution operator can be simplified: $\exp\{-iH_R\Delta t\}\times \sigma_x \exp\{-iH_R\Delta t\}\sigma_x \approx \exp\{(\lambda_n\Delta t/2)(a^2 - a^{\dagger 2})\sigma_x\}$. If we initialize the charge qubit in the $\sigma_x = 1$ state, and repeat this procedure N times, the evolution operator on the state of the resonator becomes

$$S(\kappa) = \exp\left\{\frac{\kappa}{2}(a^2 - a^{\dagger 2})\right\}, \tag{4}$$

where $\kappa = \lambda_n N\Delta t$. This is a squeezing operator on the nanomechanical resonator with squeezing parameter κ. Under the squeezing operator, a transforms to $S^\dagger(\kappa)aS(\kappa) = a\cosh\kappa - a^\dagger \sinh(\kappa)$ and it can be shown that the position uncertainty decreases exponentially [10]: $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2} = \Delta X(0)e^{-\kappa}$.

Ideally, one would like a large coupling strength in order to generate appreciable squeezing in a short period of time. For a magnetic field $B \approx 0.2$ T, $l \approx 30$ μm, $\delta X_0 \approx 5 \times 10^{-13}$ m, and a critical current of about 60 nA, the coupling strength λ_n is about 4 MHz.

Decoherence.—To study the effect of decoherence, we need to include all sources of decoherence and dissipation of both the nanoresonator and charge qubit since they are coupled. We use the master equation [10] $d\rho/dt = -i[H_{\rm sq}, \rho] + \mathcal{L}(a, \gamma_n, N_n) + \mathcal{L}(\sigma^-, \gamma_q, N_q) + \mathcal{L}(\sigma_z, \gamma_\phi, N_q)$, where ρ is the density matrix of the nanoresonator–charge-qubit system, $H_{\rm sq} = i\lambda(a^2 - a^{\dagger 2})\sigma_x/2$ is the effective squeezing Hamiltonian, $\gamma_n = \omega_0/Q$ is the decay rate of the nanoresonator determined by its quality factor Q, γ_q and γ_ϕ are the relaxation and dephasing rate of the charge qubit, N_n and N_q are the mean values of the bath quanta dependent on temperature [10], and the Liouvillian operator $\mathcal{L}(A, \gamma, N) = (1/2)\gamma(N + 1)(2A\rho A^\dagger - A^\dagger A\rho - \rho A^\dagger A) + (1/2)\gamma N(2A^\dagger \rho A - AA^\dagger\rho - \rho AA^\dagger)$. Among the various decoherence sources, the dephasing of the charge qubit is dominant [18,19]. Most nanoresonators used in experiments have frequencies of 100–250 MHz [2]. If high charging energies are chosen, we only need to bias the charge qubit slightly away from the charge degeneracy point ($|2n_g - 1| \ll 0.1$), and the dephasing time T_2 is greater than 100 ns [18]. Using the master equation we can derive equations for the expectation values of the dynamic variables of the system which are then solved. In Fig. 2(a), we plot the time dependence of the nanoresonator position uncertainty ΔX. We have used the following conservative set of experimental parameters: resonator frequency $\omega_0/2\pi = 250$ MHz, quality factor $Q = 10^4$, temperature $T = 20$ mK, squeezing parameter $\lambda = 5$ MHz, with γ_q and γ_ϕ chosen such that the relaxation time $T_1 = 1$ μs and dephasing time $T_2 = 100$ ns. Initially the nanoresonator is in a thermal equilibrium state and the charge qubit is in the $\sigma_x = 1$ state.

It is clear from Fig. 2(a) that appreciable squeezing can be generated even when the charge qubit's dephasing rate is severe (twice the squeezing parameter), indicating that the squeezing is robust against decoherence. As time progresses, the squeezing becomes less effective and eventually ΔX starts to increase. This is easy to understand from the effective Hamiltonian $i\lambda(a^2 - a^{\dagger 2})\sigma_x$ leading to (4); as the charge qubit dephases, $\langle\sigma_x\rangle$ decreases and the squeezing effect weakens and eventually disappears. Not surprisingly, the maximum squeezing achievable increases with decreasing dephasing rate, as shown in Fig. 2(b).

Measurement.—One way to measure the uncertainty in the nanoresonator's position is quantum tomography. This measures the resonator's Wigner function, which can be done by displacing the state of the harmonic oscillator, letting it interact with a two state system, and measuring the polarization of the two state system [20]. Though this method can determine all the information about the oscillator, its direct application in our solid state system is difficult. To displace the resonator's state arbitrarily, we will need couplings to both the position and momentum operators of the resonator with continuously variable relative strengths, which is not easy to realize. Also, in order to calculate ΔX, we need the Wigner function over its entire parameter space making it necessary to sweep through a large parameter space. Large displacement will take a long

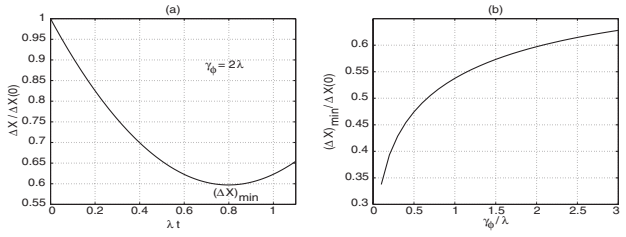

FIG. 2. (a) Time dependence of the nanoresonator position uncertainty ΔX (normalized to the initial uncertainty) under the effect of squeezing and decoherence. See text for experimental parameters and initial conditions. (b) The maximum achievable squeezing (minimum ΔX) as a function of the dephasing rate. All other parameters are the same as in (a).

time to effect and the decoherence will affect the measurement result.

Here we propose a simplified method to determine the uncertainty in the nanoresonator's position by measuring its generating function $\mathrm{Tr}(\rho e^{i\kappa X})$. We notice that $\langle X\rangle$ and $\langle X^2\rangle$ can be calculated by

$$\langle X\rangle = \mathrm{Tr}(\rho X) = -i\frac{d}{d\kappa}\,\mathrm{Tr}(\rho e^{i\kappa X})|_{\kappa=0} \tag{5}$$

and

$$\langle X^2\rangle = \mathrm{Tr}(\rho X^2) = -\frac{d^2}{d\kappa^2}\,\mathrm{Tr}(\rho e^{i\kappa X})|_{\kappa=0}. \tag{6}$$

Therefore, measuring the generating function in the vicinity of $\kappa = 0$ allows us to determine ΔX^2.

The generating function can be measured by first preparing the charge qubit in the $\sigma_z = 1$ state. Then turning on a strong coupling $\lambda X\sigma_x$ between the charge qubit and the resonator, for a time t, will cause the charge qubit states corresponding to $\sigma_x = \pm 1$ to acquire a phase shift $\mp\lambda t X$. Next, the interaction is turned off and the gate $\mathsf{H}\exp\{i(\pi/4)(\vec{n}\cdot\vec{\sigma})\}$ is applied to the charge qubit, where $\mathsf{H} = (\sigma_x + \sigma_z)/\sqrt{2}$ is the Hadamard gate, $\vec{n} = (0, -\cos\eta, \sin\eta)$ with η an angle chosen to be 0 or $\pi/2$. We then measure the polarization of the charge qubit, $\langle\sigma_z\rangle = P_{\sigma_z=1} - P_{\sigma_z=-1}$, which can be shown to equal $\mathrm{Tr}(\rho\mathrm{Re}\{e^{i\eta}e^{i\kappa X}\})$, where $\kappa = 2\lambda t$. Choosing $\eta = 0$ and $\pi/2$ then yields the real and imaginary part of the generating function, which in turn allows us to calculate ΔX^2 by Eqs. (5) and (6).

This method requires only strong linear coupling of the charge qubit to the position operator of the resonator, which as discussed before can be realized in our scheme by biasing the SQUIDs at $(n + 1/2)\Phi_0$ and the big loop at 0 flux [15,16]. We only need to measure in the vicinity of $\kappa = 0$; therefore, the measurement can be done quickly, implying less influence by decoherence. Thus, our scheme is realistic given currently available technology.

Conclusion.—We have proposed a scheme to couple a nanomechanical resonator to Josephson quantum circuits by modulating the magnetic bias of a SQUID. This allows us to realize coherent nonlinear effects on the nanoresonator, which are essential for, but so far missing in, the study of nanomechanical resonators. Though we focused on the squeezing of a nanomechanical resonator by coupling it to a charge qubit, our scheme can be easily tailored for other purposes and adapted for coupling to other Josephson quantum circuits. It can be directly extended for quantum manipulation of multiple nanoresonators, and can thus provide a practically feasible approach for unambiguous demonstration of quantum behavior in nanomechanical systems.

We noted two recent works that are related to this work [21,22]. We thank B. L. T. Plourde for valuable discussions on the circuit design and fabrication issues. This work was supported by the Packard Foundation.

[1] X. M. H. Huang *et al.*, Nature (London) **421**, 496 (2003).
[2] R. G. Knobel and A. N. Cleland, Nature (London) **424**, 291 (2003); M. D. LaHaye *et al.*, Science **304**, 74 (2004).
[3] A. D. Armour, M. P. Blencowe, and K. C. Schwab, Phys. Rev. Lett. **88**, 148301 (2002).
[4] L. Tian, Phys. Rev. B **72**, 195411 (2005).
[5] M. F. Bocko and R. Onofrio, Rev. Mod. Phys. **68**, 755 (1996).
[6] W. J. Munro *et al.*, Phys. Rev. A **66**, 023819 (2002).
[7] A. N. Cleland and M. R. Geller, Phys. Rev. Lett. **93**, 070501 (2004).
[8] A. Gaidarzhy *et al.*, Phys. Rev. Lett. **94**, 030402 (2005); See also K. C. Schwab *et al.*, Phys. Rev. Lett. **95**, 248901 (2005); A. Gaidarzhy *et al.*, Phys. Rev. Lett. **95**, 248902 (2005).
[9] Y. Makhlin, G. Schön, and A. Shnirman, Rev. Mod. Phys. **73**, 357 (2001).
[10] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1997).
[11] S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005).
[12] A. M. Kadin, *Introduction to Superconducting Circuits* (Wiley, New York, 1999). Here we assume that the self inductance of the SQUID is small and neglect its effect, which does not affect the conclusion in a qualitative way.
[13] P. Rabl, A. Shnirman, and P. Zoller, Phys. Rev. B **70**, 205304 (2004).
[14] R. Ruskov, K. Schwab, and A. N. Korotkov, Phys. Rev. B **71**, 235407 (2005).
[15] This can be realized by using multiple on-chip flux bias lines coupled to the loops in different strengths. By adjusting the currents in the bias lines, the flux biases of the loops can be controlled individually [16].
[16] B. L. T. Plourde *et al.*, Phys. Rev. B **72**, 060506(R) (2005).
[17] J. Wei and E. Norman, J. Math. Phys. (N.Y.) **4**, 575 (1963).
[18] D. Vion *et al.*, Fortschr. Phys. **51**, 462 (2003); G. Ithier *et al.*, Phys. Rev. B **72**, 134519 (2005).
[19] T. Duty, D. Gunnarsson, K. Bladh, and P. Delsing, Phys. Rev. B **69**, 140503(R) (2004).
[20] L. G. Lutterbach and L. Davidovich, Phys. Rev. Lett. **78**, 2547 (1997).
[21] L. Tian, cond-mat/0606787.
[22] E. Buks and M. P. Blencowe, Phys. Rev. B **74**, 174504 (2006).