Entangled Three-Qubit States without Concurrence and Three-Tangle

Robert Lohmayer,[1] Andreas Osterloh,[2] Jens Siewert,[3,1] and Armin Uhlmann[4]

[1]*Institut für Theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*

[2]*Institut für Theoretische Physik, Universität Hannover, D-30167 Hannover, Germany*

[3]*MATIS-INFM Dipartimento di Metodologie Fisiche e Chimiche (DMFCI), viale A. Doria 6, 95125 Catania, Italy*

[4]*Institut für Theoretische Physik, Universität Leipzig, D-04109 Leipzig, Germany*

We provide a complete analysis of mixed three-qubit states composed of a Greenberger-Horne-Zeilinger state and a *W* state orthogonal to the former. We present optimal decompositions and convex roofs for the three-tangle. Further, we provide an analytical method to decide whether or not an arbitrary rank-2 state of three qubits has vanishing three-tangle. These results highlight intriguing differences compared to the properties of two-qubit mixed states, and may serve as a quantitative reference for future studies of entanglement in multipartite mixed states. By studying the Coffman-Kundu-Wootters inequality we find that, while the amounts of inequivalent entanglement types strictly add up for pure states, this "monogamy" can be lifted for mixed states by virtue of vanishing tangle measures.

Characterizing and quantifying entanglement of multipartite mixed states is a fundamental issue in quantum information theory, both from a theoretical and a practical point of view [1–3]. Despite the large number of profound results, e.g., [4–13] even for the simplest case—the one of three qubits—there is no general solution to this problem. While the entanglement problem is well understood for pure and mixed states of two qubits [1–3,14–16], expanding our knowledge to both higher-dimensional systems and systems of more than two qubits turned out to be a formidable task. Although pure three-qubit states are well characterized by their local invariants and their nontrivial entanglement properties, similarly complete understanding of mixed three-qubit states remains elusive.

In Ref. [15] it has been worked out that mixed states are characterized in terms of the convex roof for pure-state entanglement measures. The seminal paper by Coffman *et al.* [5] provided a basis for the quantification of three-party entanglement by introducing the so-called residual tangle that led to understanding important subtleties in multipartite quantum correlations [6]. To date, there is no simple analytical method to compute the residual tangle for mixed three-qubit states. Therefore, it may be helpful to study simple cases of such mixed states which, on the one hand, allow for a complete characterization, but on the other hand involve as many features as possible that go beyond the well-known bipartite case.

To this end, we study a particular family of rank-2 mixed three-qubit states. After briefly introducing the basic concepts to describe two-qubit and three-qubit entanglement we first analyze superpositions of Greenberger-Horne-Zeilinger (GHZ) and *W* states. Then we present the results for the residual tangle of GHZ/*W* mixtures that lead us to the optimal decompositions, and the corresponding convex roofs. Surprisingly, we obtain the residual tangle not only for mixtures of GHZ and *W* states, but also for a large part

of the Bloch sphere spanned by these states. Finally, we discuss the Coffman-Kundu-Wootters (CKW) inequality. Our results provide a rich reference for further work on the quantification of multipartite entanglement.

Important concepts.—The concurrence $C(\phi_{AB})$ measures the degree of bipartite entanglement shared between the parties A and B in a pure two-qubit state $|\phi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. In terms of the coefficients $\{\phi_{00}, \phi_{01}, \phi_{10}, \phi_{11}\}$ of $|\phi_{AB}\rangle$ with respect to an orthonormal basis it is defined as

$$C = 2|\phi_{00}\phi_{11} - \phi_{01}\phi_{10}|. \tag{1}$$

The concurrence is maximal for Bell states like $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and vanishes for factorized states.

The three-tangle (or residual tangle) $\tau_3(\psi)$, a measure for three-party entanglement in the three-qubit state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ has been introduced in Ref. [5]. It can be expressed by using the wave function coefficients $\{\psi_{000}, \psi_{001}, \ldots, \psi_{111}\}$ as

$$\tau_3 = 4|d_1 - 2d_2 + 4d_3|,$$
$$d_1 = \psi_{000}^2\psi_{111}^2 + \psi_{001}^2\psi_{110}^2 + \psi_{010}^2\psi_{101}^2 + \psi_{100}^2\psi_{011}^2,$$
$$d_2 = \psi_{000}\psi_{111}\psi_{011}\psi_{100} + \psi_{000}\psi_{111}\psi_{101}\psi_{010}$$
$$\quad + \psi_{000}\psi_{111}\psi_{110}\psi_{001} + \psi_{011}\psi_{100}\psi_{101}\psi_{010}$$
$$\quad + \psi_{011}\psi_{100}\psi_{110}\psi_{001} + \psi_{101}\psi_{010}\psi_{110}\psi_{001},$$
$$d_3 = \psi_{000}\psi_{110}\psi_{101}\psi_{011} + \psi_{111}\psi_{001}\psi_{010}\psi_{100}. \tag{2}$$

We mention that this definition of the three-tangle coincides with the modulus of a so-called hyperdeterminant which was introduced by Cayley [17,18]. For the GHZ state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{3}$$

the three-tangle becomes maximal: $\tau_3(\text{GHZ}) = 1$, and it vanishes for any factorized state. Remarkably, there is a class of entangled three-qubit states for which τ_3 vanishes [6]. This class is represented by the $|W\rangle$ state

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle). \quad (4)$$

Now consider, e.g., mixed two-qubit states ω with their decompositions

$$\omega = \sum_j p_j \pi_j, \qquad \pi_j = \frac{|\phi_j\rangle\langle\phi_j|}{\langle\phi_j|\phi_j\rangle}. \quad (5)$$

The concurrence of the mixed two-qubit state ω is defined as the average pure-state concurrence minimized over all decompositions

$$C(\omega) = \min \sum p_j C(\pi_j). \quad (6)$$

which is also called a convex-roof extension [3,15,19]. The mixed-state three-tangle is defined analogously. A decomposition that realizes the minimum of the respective function, is called optimal.

In this Letter we study the three-tangle of rank-2 mixed three-qubit states

$$\rho(p) = p\pi_{\text{GHZ}} + (1 - p)\pi_W \quad (7)$$

with $\pi_{\text{GHZ}} = |\text{GHZ}\rangle\langle\text{GHZ}|$ and $\pi_W = |W\rangle\langle W|$ (notice that $\langle\text{GHZ}|W\rangle = 0$).

Superpositions of GHZ and W states.—It is well known that from the decomposition of a rank-n density matrix $\rho = \sum_{j=1}^{n} p_j |j\rangle\langle j|$ into its eigenstates $\{|j\rangle\}$, any other decomposition $\rho = \sum |\chi_l\rangle\langle\chi_l|$ of length $m \geq n$ can be ob-
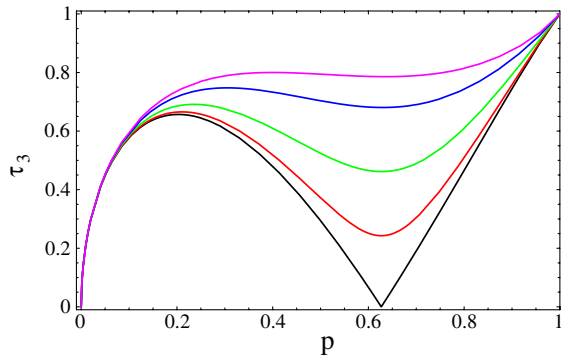

FIG. 1 (color online). The three-tangle of the states $|Z(p, \varphi)\rangle$ in Eq. (8) for various values of $\varphi = \gamma \cdot \frac{2\pi}{3}$ (from top to bottom: $\gamma = 1/2, 1/3, 1/5, 1/10, 0$). The three-tangle is periodic in φ with a period of $2\pi/3$. Further, $\tau_3(Z)$ vanishes trivially for zero GHZ component, but also for $\varphi = 0$ and nonzero GHZ amplitude $p_0 = \frac{4\sqrt[3]{2}}{3+4\sqrt[3]{2}} \approx 0.627$. This is in striking contrast to the two-qubit case where, in a superposition of a Bell state with an orthogonal unentangled state, the concurrence equals the weight of the Bell state [25], that is the entanglement remains "untouched".

tained with a unitary $m \times m$ matrix U_{lj} via $|\chi_l\rangle = \sum_{j=1}^{m} U_{lj}(\sqrt{p_j}|j\rangle)$. Hence, the vectors of any decomposition of our states $\rho(p)$ are linear combinations of $|\text{GHZ}\rangle$ and $|W\rangle$. Therefore, we first analyze the three-tangle of the states

$$|Z(p, \varphi)\rangle = \sqrt{p}|\text{GHZ}\rangle - e^{i\varphi}\sqrt{1 - p}|W\rangle. \quad (8)$$

With Eq. (2) we obtain (see also Fig. 1)

$$\tau_3[Z(p, \varphi)] = \left| p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1 - p)^3}e^{3i\varphi} \right|. \quad (9)$$

We notice that $\tau_3(Z)$ has a nontrivial zero ($\varphi = 0$)

$$p_0 = \frac{4\sqrt[3]{2}}{3 + 4\sqrt[3]{2}} = 0.626\,851\ldots.$$

Mixtures of GHZ and W states.—An immediate consequence of the existence of a finite p_0 with $\tau_3[Z(p_0)] = 0$, together with the permutation symmetry of the states, is that the mixed-state three-tangle $\tau_3[\rho(p)] = 0$ for $p = p_0$ as well as for all $0 \leq p \leq p_0$. For $p = p_0$ we have the optimal decomposition

$$\rho(p_0) = \frac{1}{3}(|Z_0^0\rangle\langle Z_0^0| + |Z_0^1\rangle\langle Z_0^1| + |Z_0^2\rangle\langle Z_0^2|),$$

$$|Z_0^j\rangle = \sqrt{p_0}|\text{GHZ}\rangle - e^{(2\pi i/3)j}\sqrt{1 - p_0}|W\rangle. \quad (10)$$

For $p < p_0$ there is a decomposition with vanishing three-tangle of the form $\rho(p) = \frac{p}{p_0}\rho(p_0) + (1 - \frac{p}{p_0})\pi_W$. For $p > p_0$, all states $\rho(p)$ have nonvanishing three-tangle.

The decomposition in Eqs. (10) provides a trial decomposition $\mathcal{S}$ for $\rho(p > p_0)$ given by $\mathcal{S} = \{\sqrt{\frac{1}{3}}|Z^0\rangle, \sqrt{\frac{1}{3}}|Z^1\rangle, \sqrt{\frac{1}{3}}|Z^2\rangle\}$ with $|Z^j\rangle = |Z(p, \frac{2\pi}{3}j)\rangle$. Its three-tangle

$$g_I(p) = p^2 - \frac{8\sqrt{6}}{9}\sqrt{p(1 - p)^3}, \qquad p \geq p_0 \quad (11)$$

provides an upper bound for $\tau_3[\rho(p > p_0)]$: as the function $g_I(p)$ is not convex in the entire interval $p_0 \leq p \leq 1$, it cannot represent the three-tangle of $\rho(p)$ for all $p \in [p_0, 1]$.

In order to test this upper bound, we have performed extensive numerical studies [the numerical method can be tested with the two-qubit case as well as with states $\rho(p \leq p_0)$ with vanishing three-tangle]. According to Caratheodory's theorem, for rank-2 states four vectors are sufficient to minimize the three-tangle. Hence we need to investigate decompositions with 2, 3, or 4 vectors only.

Our numerical results indicate that $\mathcal{S}$ is indeed an optimal decomposition for values of p close to p_0. For larger p, however, we propose the four-vector decomposition $\mathcal{T}$ that consists of three vectors analogous to $\mathcal{S}$ and a *pure* GHZ state:

$$\rho(p) = (1-b)|\mathrm{GHZ}\rangle\langle\mathrm{GHZ}| + \frac{b}{3}\sum_{j=0}^{2}|Z^j(a)\rangle\langle Z^j(a)|,$$

where $|Z^j(a)\rangle \equiv |Z(a, \frac{2\pi}{3}j)\rangle$. The coefficients a and b have to be determined as functions of p. It turns out that the decomposition $\mathcal{S}$ is optimal for $p_0 \le p \le p_1 = \frac{1}{2} + \frac{3}{310} \times \sqrt{465} \equiv 0.708\,68\ldots$. For $p \in [p_1, 1]$ the decomposition $\mathcal{T}$ has the lowest average three-tangle, given by the linear function

$$g_{II}(p) = 1 - (1-p)\left(\frac{3}{2} + \frac{1}{18}\sqrt{465}\right). \tag{12}$$

This is in complete agreement with the numerical results.

Convex roofs.—With these findings we can elucidate the affine regions (the convex roofs) of the three-tangle for any mixture of GHZ and W states. To this end, imagine the two-dimensional space spanned by the GHZ and the W state represented by a Bloch sphere with the GHZ state at its north pole and the W state at the south pole (cf. Fig. 2). All the states $|Z(p, \varphi)\rangle$ are located at the unit sphere, with azimuth φ and p the distance from the south pole along the north-south line (i.e., $p = 0$ is the south pole, and $p = 1$ is the north pole). The states with zero three-tangle are represented by the simplex S_0 with corners W, Z_0^0, Z_0^1, Z_0^2.

Now consider planes $P(p)$ parallel to the ground plane of the simplex (i.e., the plane containing the triangle Z_0^0, Z_0^1, Z_0^2) and intersection point p with the north-south line. For $p_0 \le p \le p_1$ the three-tangle is constant in a triangle that has its corners at the intersection points of the plane $P(p)$ and the meridians through Z_0^0, Z_0^1, and Z_0^2 (see Fig. 2).

For $p_1 \le p \le 1$ we have another simplex S_1 with affine three-tangle. The ground plane of this simplex is formed by the last of the "leaves" (at $p = p_1$) described above, and the top corner is the GHZ state. That is, in each plane parallel to the ground plane of this simplex the three-tangle
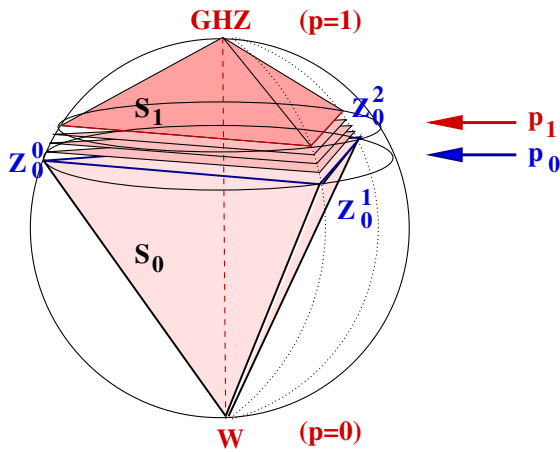
is constant. The three-tangle of any point inside S_1 (with distance p' from the south pole, i.e., the GHZ weight equals p') is a convex combination $\alpha g_{II}(p_1) + \beta \mathbf{1}$ of $g_{II}(p_1)$ (the three-tangle in the ground plane of S_1) and of 1, the value for the GHZ state. The coefficients are $\alpha = \frac{1-p'}{1-p_1}$ and $\beta = \frac{p'-p_1}{1-p_1}$. This completes the characterization of the mixed states $\rho(p)$.

CKW inequality.—Given a whole family of mixed three-qubit states with the corresponding three-tangle, one might like to check the CKW relations [5]. To this end, we consider pure three-qubit states $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and introduce the reduced two-qubit density matrices $\rho_{AB} = \mathrm{Tr}_C(|\psi\rangle\langle\psi|)$, $\rho_{AC} = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$, and the reduced density matrix of the first qubit $\rho_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|)$. For pure states, one has the "monogamy relation" $4\det(\rho_A) = C(\rho_{AB})^2 + C(\rho_{AC})^2 + \tau_3(\psi)$, that is, the entanglement of qubit A can be either bipartite entanglement with B or C (concurrence), or three-party entanglement with B and C (three-tangle). For mixed three-qubit states ρ, Coffman *et al.* found the inequality

$$4\min[\det(\rho_A)] \ge C(\rho_{AB})^2 + C(\rho_{AC})^2. \tag{13}$$

Here, also the one-tangle has to be minimized for all possible decompositions of ρ. The reduced two-qubit density matrices are defined analogously as $\rho_{AB} = \mathrm{Tr}_C(\rho)$, $\rho_{AC} = \mathrm{Tr}_B(\rho)$, $\rho_A = \mathrm{Tr}_{BC}(\rho)$. We refer to Eq. (13) as the "CKW inequality".

Because of the invariance of $\rho(p)$ with respect to qubit permutations, we can consider the CKW inequality with respect to the first qubit, without loss of generality. It is straightforward to compute the concurrences:

$$C_{AB}^2 + C_{AC}^2 = 2\left(\max\left[0, \frac{2}{3}(1-p) - \sqrt{\frac{p}{3}(2+p)}\right]\right)^2. \tag{14}$$

This is a monotonously decreasing function for $p \in [0, 1]$ that vanishes for $p_C = 7 - \sqrt{45} \approx 0.292$. For the minimum one-tangle we obtain

FIG. 2 (color online).   Bloch sphere for the two-dimensional space spanned by the GHZ state and the W state. The simplex S_0 contains all states with zero three-tangle. The leaves between p_0 and p_1 represent sets of constant three-tangle, and in the simplex S_1 the three-tangle is affine (for further explanation, see text).
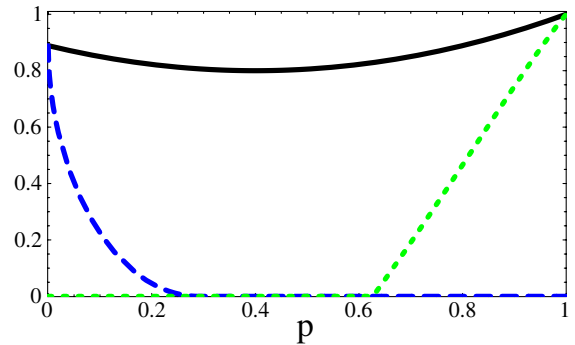

FIG. 3 (color online).   Plot of the one-tangle (solid line), the sum of squared concurrences $C(\rho_{AB})^2 + C(\rho_{AC})^2$ (dashed line) and three-tangle (dotted line) in $\rho(p)$ as a function of p. Clearly, the CKW inequality (13) is satisfied.

$$4 \min[\det(\mathrm{Tr}_{BC}\rho(p))] = \tfrac{1}{9}(5p^2 - 4p + 8). \qquad (15)$$

In order to interpret these results, we plot the one-tangle, the sum of squared concurrences, and the three-tangle, see Fig. 3. The one-tangle is always larger than the sum of squared concurrences and three-tangle. In particular, there is a region $p_C \le p \le p_0$ where there is quite substantial one-tangle, but there is *zero* concurrence and *zero* three-tangle in the state $\rho(p)$. Our interpretation of this result is that it is not predetermined whether the entanglement contained in $\rho(p)$ is bipartite entanglement or three-way entanglement. According to the choice of decomposition the entanglement in $\rho(p)$ can be represented either in terms of bipartite correlations, or as three-partite correlations, respectively. This is a curious fact if we recall that the GHZ and the *W* state are locally inequivalent. Apparently, for mixed states, there is no strict monogamy as for pure states. As opposed to this, for "less mixed states"—i.e., for $p < p_C$ and for $p > p_0$—the entanglement stems (mandatorily) to a large extent either from bipartite correlations, or from three-way correlations.

Conclusion.—Summarizing, we have studied the entanglement properties of the family of mixed three-qubit states $\rho(p)$ [cf. Eq. (7)]. As opposed to the two-qubit case (where one has a huge variety of optimal decompositions [14,15]), for our three-qubit states $\rho(p)$ there appears to be only a single type of optimal decomposition (apart from phase rotations). Further, while a two-qubit state of rank n has always an optimal decomposition of length n, this does not hold for the three-qubit states $\rho(p)$. Remarkably, we have obtained the convex roof of the three-tangle not only for mixtures as in Eq. (7), but for a considerable part of the Bloch sphere (cf. Fig. 2). We mention that, if any of these density matrices is convexly combined with an *arbitrary* three-qubit density matrix, our results provide a nontrivial upper bound for the three-tangle of the resulting state.

Moreover, we have found that for the entanglement contained in multipartite mixed states there is no strict monogamy; i.e., it can be represented by different types of locally inequivalent quantum correlations.

Finally, we would like to point out that from our results we may draw an important conclusion for arbitrary rank-2 density matrices $\rho = p|1\rangle\langle 1| + (1 - p)|2\rangle\langle 2|$, of three qubits. Also in the general case, there will exist a simplex S_0 which makes it possible to decide on analytical grounds whether or not a rank-2 state has vanishing three-tangle. This can be seen as follows.

Consider the three-tangle of the state $|\psi\rangle = |1\rangle + z|2\rangle$ with a complex number z (normalization is irrelevant here). The condition for the three-tangle of this state to be zero [analogously to Eq. (9)] is given by a polynomial equation of 4th degree in z. Hence, there are four (in general different) pure states with vanishing three-tangle that define the simplex S_0 in the corresponding Bloch sphere which contains those density matrices whose three-tangle is zero. Thus, the three-tangle of ρ vanishes iff it belongs to S_0.

The implications of the concept of the simplex S_0 are reaching even further. It can be generalized for multipartite entanglement monotones of pure states [20–22]. A method to explicitly write N-qubit entanglement monotones ($N \ge 3$) as certain expectation values of antilinear operators has been presented in Refs. [23,24]. Let us consider an arbitrary rank-2 N-qubit density matrix. In complete analogy to what was said above, one can find the zeros of such N-tangles on the Bloch sphere which then define a "zero-polyhedron" (rather than a simplex) that contains all the density matrices with vanishing N-tangle.

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