# Nonlinear Quantum Shock Waves in Fractional Quantum Hall Edge States

E. Bettelheim,[1] Alexander G. Abanov,[2] and P. Wiegmann[1,*]

[1]*James Frank Institute, University of Chicago, 5640 S. Ellis Avenue, Chicago, Illinois 60637, USA*
[2]*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794-3800, USA*
(Received 27 June 2006; published 15 December 2006)

Using the Calogero model as an example, we show that the transport in interacting nondissipative electronic systems is essentially nonlinear and unstable. Nonlinear effects are due to the curvature of the electronic spectrum near the Fermi energy. As is typical for nonlinear systems, a propagating semiclassical wave packet develops a shock wave at a finite time. A wave packet collapses into oscillatory features which further evolve into regularly structured localized pulses carrying a fractionally quantized charge. The Calogero model can be used to describe fractional quantum Hall edge states. We discuss perspectives of observation of quantum shock waves and a direct measurement of the fractional charge in fractional quantum Hall edge states.

DOI: 10.1103/PhysRevLett.97.246401 PACS numbers: 71.10.Pm, 02.30.Ik, 73.43.Lp

It is commonly assumed that a small and smooth perturbation of electronic systems remains so in the course of evolution. This assumption justifies the linear response theory of transport. Although it is true for dissipative electronic systems, like metals, it fails for dissipationless quantum liquids.

In this Letter we argue that in such electronic systems, commonly known in one spatial dimension, the transport is essentially nonlinear, featuring an unstable singular behavior, "a gradient catastrophe". An arbitrarily small and smooth disturbance of electronic density and momentum (a wave packet) will inevitably form a shock wave evolving to shot-noise-like oscillatory features.

The inevitable emergence of shock waves in dissipationless quantum liquids is a consequence of two general properties: (i) being many body systems they are described by quantum hydrodynamics, i.e., solely by density $\hat{\rho}(x) = \sum_i \delta(x - x_i)$ and velocity operators $[\hat{\rho}(x), \hat{v}(y)] = -i\frac{\hbar}{m}\delta'(x - y)$; (ii) the liquids are Galilean invariant and compressible. These properties alone yield equations of quantum hydrodynamics [1]

$$\text{continuity equation}: \dot{\hat{\rho}} + \nabla(\hat{\rho}\,\hat{v}) = 0, \tag{1}$$

$$\text{Euler equation}: \dot{\hat{v}} + \nabla\left(\frac{\hat{v}^2}{2} + w\right) = 0, \tag{2}$$

where the enthalpy $w[\hat{\rho}]$ is an interaction dependent function of the density. In this approach excitations appear as solitons of nonlinear quantum fields $\hat{\rho}(x)$ and $\hat{v}(x)$. Direct derivations of these equations starting from many body Hamiltonians are available for the Luttinger liquid [2] and Calogero model [3].

Although the linear approximation correctly captures on-shell physics at $\omega \approx v_s k$, it misses important off-shell phenomena which are the most visible in nonequilibrium processes. Shock-wave instabilities discussed in this Letter are one such phenomena. They emerge as nonperturbative effects of the nonlinear terms in (1) and (2). The early stage of a shock wave is not sensitive to the type of interaction, or even to its very existence, contrary to the later stage, occurring after the collapse.

We will demonstrate this phenomenon on the example of the Calogero model. Being interesting in its own right, the chiral sector of the Calogero model captures the physics of some (Abelian) fractional quantum Hall edge states. Moreover, our estimates indicate that it is feasible to observe the onset of a shock wave in realistic confined states of quantum well heterostructures.

We will show that any initially smooth semiclassical excitation collapses into oscillatory features which further evolve into regularly structured localized pulses carrying a fractionally quantized charge—*soliton trains*. Experimental detection of these pulses would be a direct observation of the fractional charge of "flyout" excitations.

*Calogero-Sutherland model.*—Defined on a circle by

$$\mathcal{H} = -\sum_{j=1}^{N}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_j^2} + \sum_{k\neq j=1}^{N}\frac{\pi^2}{2L^2}\frac{g}{\sin^2(\frac{\pi}{L}(x_j - x_k))} \tag{3}$$

occupies a special place in 1D quantum physics. The singular interaction changes the statistics of particles from 1 (Fermi statistics) to $\lambda$, while excitations carry statistics of $1/\lambda$ (see, e.g., [4]). All these match the phenomenological description of the (Abelian) fractional Hall edge states with a fractional filling $1/\lambda$ [5] and justify the use of the Calogero model for edge states. Here $\lambda$ is a dimensionless coupling constant $g = \frac{\hbar^2}{m}\lambda(\lambda - 1)$. We also set $a = (\lambda - 1)/\lambda$, $\kappa = 2\pi\lambda\hbar/m$, and rely on the results of [6].

*Evolution of a semiclassical state.*—Consider an evolution of an initial state—a wave packet $|\Psi\rangle$, of a "large" size $l \gg k_F^{-1}$ as in Fig. 1(a). The state is assumed to be semiclassical, i.e., it involves many particles, and is such that its Wigner's function $W(x, p) = \langle\Psi|e^{(i/\hbar)(\hat{P}x + p\hat{X})}|\Psi\rangle$ has a well-defined classical limit. This means that $W(x, p)$ exponentially vanishes outside of a certain support in phase space. Semiclassical hydrodynamic modes can be viewed as area-preserving waves of the boundary of the

 

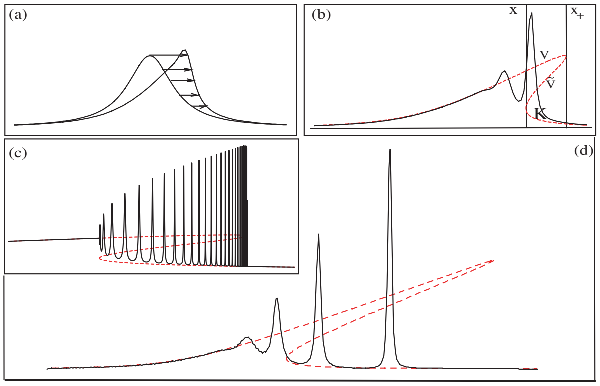


FIG. 1 (color online). Numerical solution of BO Eq. (8). (a) The initial Lorentzian profile containing 7 solitons is shown at time $t = 0.9t_c$. (b) The wave after a shock is shown together with the overhanging solution of Riemann Eq. (5) (red, dashed) at $t = 2.7t_c$. Vertical lines mark the trailing and the leading edges. The moduli $v$, $\tilde{v}$, $K$ of the one-phase solution (9) are assigned to different branches of the multivalued solution of (5). (d) The solutions of BO and Riemann's equations are shown at $t = 11t_c$. (c) Whitham modulated wave represents an approximate soliton train behavior for an initial Lorentzian shape of a larger area.

support. We study the time evolution of the average density $\rho(x, t) = \int W(x, p, t)dp = \langle\Psi|\hat{\rho}|\Psi\rangle$. These states can be created by an act of a classical instrument.

First, we argue that in an *interacting* 1D system like (3) the evolution of a semiclassical state is determined only by semiclassical hydrodynamic variables—the density $\rho(x, t)$ and velocity $v(x, t)$. In other words, the semiclassical density and velocity obey a set of closed hydrodynamic equations, which do not depend on $\hbar$.

*Quantum hydrodynamics.*—To study the dynamics of (3) and similar problems, it is productive to represent a quantum many body problem as quantum hydrodynamics (1) and (2). The Calogero model has been written in terms of hydrodynamic variables in Refs. [6–9]. In this case the energy per particle in units of $\lambda\kappa/2\pi$ is

$$\epsilon = \frac{1}{6}(\pi\hat{\rho})^2 + \frac{a^2}{8}(\nabla \ln\hat{\rho})^2 + \frac{1}{2}\pi a\nabla\hat{\rho}_H, \qquad (4)$$

where $\hat{\rho}_H = \int_0^L \frac{dx'}{L}\hat{\rho}(x')\cot\frac{\pi}{L}(x' - x)$ is a Hilbert transform of the density. The first term in (4) is the Fermi energy. It is already present in the free Fermi gas ($\lambda = 1$). The last two terms being absent at $\lambda = 1$ reflect the interaction. The energy yields for the enthalpy $w = (\kappa/2\pi)^2\delta(\rho\epsilon)/\delta\rho$ in the Euler Eq. (2).

*Sound waves, nonlinear effects, and dispersion.*—In the limit of small deviations of density $\delta\rho = \rho - \rho_0 \ll \rho_0$, and velocity $\delta v = v - v_0 \ll v_0$ from their mean values $\rho_0$ and $v_0$, Eqs. (1) and (2) may be linearized. If in addition gradients are small $a\frac{\nabla\delta\rho}{\delta\rho} \ll \rho_0$ so that the two last terms in (4) can be dropped, one obtains the familiar "linear bosonization", where density and velocity waves move to the right (left) with Fermi velocities $v_F = v_0 \pm \frac{\kappa}{2}\rho_0$ without dispersion. Introducing left and right moving fields $u = \delta v \pm \frac{\kappa}{2}\delta\rho$ the linear hydrodynamics reads $\dot{u} + v_s\nabla u = 0$, where $v_s = v_F$ are velocities of sound.

In the Galilean frame moving with variable velocity $u$, one recovers nonlinear effects

$$\dot{u} + u\nabla u = 0. \qquad (5)$$

This is the quantum Riemann equation of 1D compressible hydrodynamics [10]. The nonlinear term is traced to Galilean invariance and curved electronic spectrum. The equation states that the wave $u(x, t)$ moves with a velocity which is itself given by $u(x, t)$. Contrary to a linear spectrum, when a packet moves with the velocity of sound, and does not change its shape, the nonlinearity pushes the denser part forward.

Equation (5) is equivalent to (1) and (2) under the assumption of small gradients $a\nabla\delta\rho \ll \delta\rho^2$ when the two last terms in (4) are small. However this assumption almost never holds. It fails for any small and smooth initial condition when $u(x)\nabla u(x) < 0$. This is seen from the implicit solution of (5) due to Riemann: $u = f(x - ut)$, where $f(x) = u(x, 0)$ is the initial state. The wave packet steepens [Fig. 1(a)] and finally overturns at some time, $t_c$, where the formal solution becomes multivalued [10], and unphysical [Figs. 1(b)–1(d)].

The time $t_c$ is roughly the time for the top of the wave packet to travel a distance of its size $l$. Since the excess velocity is $u$, an overhang occurs at a time $t_c \sim l/(\delta v \pm \frac{\kappa}{2}\delta\rho)$. At this time the curvature of the spectrum becomes the dominant factor. Obviously, a smooth and small packet remains in the linear regime longer, but its life is finite. In the vicinity of the overhang the classical Riemann equation becomes invalid. A shock must be regularized either by quantum corrections at $\lambda = 1$, or, in the interacting case, by the neglected gradient terms.

*Dispersive regularization—the role of interaction.*—The role of dispersion changes in the presence of interaction. The latter gives higher gradient corrections to the Euler equation. In the linear regime they are small, but in the shock-wave regime, they stabilize growing gradients. This mechanism is called dispersive regularization, and is well known in the theory of nonlinear waves [11]. Stabilization occurs when dispersive and nonlinear terms in (4) become of same order

$$a\nabla\delta\rho \sim \delta\rho^2. \qquad (6)$$

This condition determines the smallest scale of oscillations occurring after a shock wave. It is $\Delta l \sim a/\delta\rho$.

This is an important result. Rising gradients are bounded by a scale exceeding the Fermi length $k_F\Delta l \gg 1$.

As a consequence, a semiclassical wave packet of interacting 1D fermions remains semiclassical even after entering a shock wave regime. Its evolution is described by classical nonlinear hydrodynamics. We are able to replace operators $\hat{\rho}$, $\hat{v}$ by their classical values $\rho$, $v$.

Condition (6) emphasizes the role of interaction. In its absence ($\lambda = 1$) the gradient catastrophe is stabilized only by quantum effects and makes a semiclassical description nonvalid at $t \sim t_c$.

*Chiral case.*—A generic wave packet will be separated into two parts—left and right modes moving away with sound velocities. The physics of shock waves is featured in each chiral part, so we can treat them separately. Also application to the edge state requires consideration of only one chiral sector. Specifics of the Calogero model allows us to separate the chiral sectors exactly, by choosing initial conditions as $\mathrm{v} = \frac{\kappa}{2}\rho$, so that the motion is only (right) chiral. Here, $\mathrm{v} = v - a\frac{\kappa}{4\pi}\nabla(\log\rho)_H$ [12]. Under this condition two Eqs. (1) and (2) become identical

$$\dot{\rho} + \frac{\kappa}{2}\nabla\left(\rho^2 + \frac{a}{2\pi}\rho\nabla(\log\rho)_H\right) = 0. \tag{7}$$

*Benjamin-Ono equation.*—Equation (7) admits further simplification relevant for the physics of shock waves. The semiclassical nature of the initial wave packet insures that deviations of density from the mean value, $\rho_0$ are small. Let us choose the chiral case, and a frame moving with the velocity $v_s = \kappa\rho_0$. Then keeping gradient terms in (7) of the lowest order, we linearize the dispersion term $\rho\nabla(\log\rho)_H \approx \nabla(\delta\rho)_H$. As a result, we obtain the celebrated Benjamin-Ono (BO) equation

$$\dot{u} + u\nabla u + a\frac{\kappa}{4\pi}\nabla^2 u_H = 0, \qquad u = \kappa\delta\rho. \tag{8}$$

In hydrodynamics it describes interface waves in a deep stratified fluid [13]. Equations (1) and (2), having a similar structure but capturing both chiral sectors, were called the Benjamin-Ono equation on the double (DBO) in [6]. Both Eqs. (7) and (8) are quantum. The field $u$ in the quantum BO equation is the gradient of a canonical real Bose field. The equation is an extension of the theory of edge states beyond the linear approximation.

*Semiclassical hydrodynamic equations, solitons.*—Below we study a semiclassical version of quantum hydrodynamics. Because of the nonlinear nature of the quantum equations, the semiclassical limit is more subtle than merely treating quantum operators as classical fields. In [6] we argued that the semiclassical limit describes the collective motion of excitations (quasiholes) with charge $1/\lambda$. This amounts to replacing the parameter $a$ in (1), (2), (7), and (8), by the charge of excitation $1/\lambda$. Indeed, one may notice that the parameter $a$ is the charge of solitons of the classical nonlinear field $u$.

Solitons and more general multiphase solutions of BO are known [14,15]. We have found general multiphase solutions and their Whitham modulations for more general DBO Eqs. (1) and (2). We present these results elsewhere [16]. Here we restrict ourselves to a simplified description of a shock wave in the most physically relevant limit when BO (8) holds. To this end we need to know only one-phase and one-soliton solutions.

The one-phase solution (compare with [17]) reads

$$\delta\rho - \overline{\delta\rho} = \frac{1}{\pi\lambda}\,\mathrm{Im}\partial_x \log\left(1 - \sqrt{\frac{v-K}{\tilde{v}-K}}e^{(i/\hbar)\theta(x,t)}\right), \tag{9}$$

where $\theta(x,t) = k(x - Vt)$, $k = m(v - \tilde{v})$, $V = \frac{1}{2}(v + \tilde{v})$, and $\overline{\delta\rho} = \frac{2}{\kappa}(K + \frac{k}{m})$ are the phase, the wave number, the velocity and the mean density. Parameters $v$, $\tilde{v}$, and $K$ are the moduli of the solution.

The one-soliton solution appears as a long wave limit $k \to 0$ of (9)

$$\delta\rho - \overline{\delta\rho} = \frac{1}{\pi\lambda}\,\mathrm{Im}\frac{1}{x - Vt - i\frac{\hbar}{2m(V - \frac{\kappa}{2}\overline{\delta\rho})}}. \tag{10}$$

The amplitude, $\frac{4(V-\frac{\kappa}{2}\overline{\delta\rho})}{\kappa}$, and the width, $\frac{\hbar}{2m(V-\frac{\kappa}{2}\overline{\delta\rho})}$ of the soliton are determined by the excess velocity $V - \frac{\kappa}{2}\overline{\delta\rho}$. The charge of the soliton is $1/\lambda$.

*Whitham modulation.*—Although Eqs. (1), (2), (7), and (8), are integrable, only quasiperiodic solutions enjoy explicit formulas [16]. Solutions with generic initial data cannot be found explicitly. However, in many physically motivated cases, the solutions are well approximated by slowly modulated waves. There it is assumed that the moduli of a quasiperiodic solution (in the case of the one-phase solution (9) the moduli are $v$, $\tilde{v}$, $K$) also depend on space and time but in a slow manner. They do not change much during a period of oscillation. If the scales of oscillations and modulations are separated, the Whitham theory provides equations for the space-time dependence of the moduli [18].

Shock waves are the most spectacular application of the Whitham theory. It has been developed in a seminal paper [19] on the example of the Korteweg-de Vries equation. The idea of the method is the following. Outside the shock-wave regime a smooth initial wave remains smooth, and one legitimately neglects the dispersion term, keeping only nonlinear terms. The equation obtained in this manner is the Riemann Eq. (5). Its solution with initial condition $u(x,0) = f(x)$ reads $u^{(0)}(x,t) = f(x - u^{(0)}(x,t)t)$. By using the superscript we indicate that this is not the oscillatory part of the solution. A solution of (5) always overturns, and, typically becomes a three-valued function in the interval $x_-(t) < x < x_+(t)$ [Fig. 1(b)], where the leading and trailing edges $x_\pm(t)$ are determined by the condition $\partial_x u^{(0)} = \infty$. Let us order the branches as $u_1^{(0)} > u_2^{(0)} > u_3^{(0)}$.

In the three-valued region the dispersion is important. It replaces a nonphysical overhang by modulated oscillations. The latter are given by the Whitham's modulated solution. To leading order one chooses a modulated one-phase solution. We call it $u^{(1)}(x,t)$ to emphasize that this solution has only one fast harmonic. It is glued with $u^{(0)}$ at $x_\pm(t)$: $u_1^{(0)}(x_-) = u^{(1)}(x_-)$, $u_3^{(0)}(x_+) = u^{(1)}(x_+)$.

A modulated one-phase solution is given by the formula (9), where three moduli $v$, $\tilde{v}$, $K$ are smooth functions of space-time, and the phase $\theta(x, t)$ is replaced by a modulated phase $\Theta(x, t)$ found from the Whitham equations $\nabla\Theta = m(v - \tilde{v})$, $\dot{\Theta} = \frac{m}{2}(v^2 - \tilde{v}^2)$. The Whitham equations for the moduli have again the form of Riemann Eqs. (5) $\dot{v} + v\nabla v = 0$, $\dot{\tilde{v}} + \tilde{v}\nabla\tilde{v} = 0$, $\dot{K} + K\nabla K = 0$. The boundary data for the moduli are chosen such that $u^{(1)}$ stops to oscillate at the gluing points $x_\pm$. That happens (i) when $v \to \tilde{v}$ at $x = x_+(t)$—there $u^{(1)} \to K$, and (ii) when $\tilde{v} \to K$, at $x = x_-(t)$—there $u^{(1)} \to \tilde{v}$. The gluing conditions lead to an especially simple result for the moduli of BO equation [15,20]: $v = u_1^{(0)}(x, t)$, $\tilde{v} = u_2^{(0)}(x, t)$, $K = u_3^{(0)}(x, t)$.

*Shock waves.*—The space-time dependence of the moduli approximately determines the entire evolution of a wave packet. We summarize some features which are independent of the initial shape. Let us choose a frame moving with the sound velocity. An initially smooth bump with a height $\delta\rho$ and a width $l$ tends to produce an overhang. At the time $t_c \sim l/\kappa\delta\rho$ the bump starts to produce oscillations. The oscillations fill the growing interval $x_-(t) < x < x_+(t)$. The leading edge moves with a velocity exceeding sound. At large time $x_+ \sim \kappa\delta\rho t$.

The amplitude of oscillations is zero at the trailing edge and grows towards the leading edge. Furthermore, the period of the oscillations also grows. As a result, near the leading edge the oscillatory pattern resembles a collection of individual localized traveling pulses (solitons)—a soliton train. The excess velocity, $V - \frac{\kappa}{2}\overline{\delta\rho}$, of the leading soliton is $\kappa\delta\rho$, which also determines the amplitude $4\delta\rho$. The height of the leading pulse is 4 times that of the initial bump. Since each pulse carries a quantized charge $1/\lambda$, the width of leading pulses, $(4\pi\delta\overline{\delta\rho}\lambda)^{-1}$ stays constant, while the distance between them is growing.

*Observation of shock waves and direct measurement of a fractional charge.*—The fractional Hall edge seems to be the best electronic system to observe quantum shock waves. Below we discuss the feasibility of an experimental measurement, probing nonlinear and dispersive effects at the edge of a quantum Hall system. This measurement would reveal the fractionally charged excitations at the edge.

The time scales of the shock waves depend on parameters of the linearized theory of the edge—the sound velocity and the compressibility of the chiral boson. Although these quantities have never been measured directly, one can suggest estimates [21]. We estimate the sound velocity on the edge as $v_s \sim 3 \times 10^4$ m/s [22]. An electronic density in the bulk $10^{11}$ cm$^{-2}$ gives an estimate for the 1D density on the edge $\rho_0 \sim 3 \times 10^7$ m$^{-1}$. It gives $\kappa = v_s/\rho_0 \sim 10^{-3}$ m$^2$/s, and, at $\lambda = 3$ estimates an effective mass $m$ to be about 30 electron band masses in GaAs and the "Fermi" time $\tau_F = (\kappa\rho_0^2)^{-1}$ to be ~1 ps.

A wave packet of width $l \sim 10^2\rho_0^{-1} \sim 1$ $\mu$m and of height $\delta\rho \sim 10^{-1}\rho_0$ carrying about 10 electrons, develops a shock wave at time $t_c \sim \tau_F(\rho_0 l)\frac{\rho_0}{\delta\rho} \sim 1$ ns. During the time $t_c$ the wave packet crosses a distance $v_s t_c \sim 30$ $\mu$m. This scale is much smaller than a size of a typical heterostructure (~$10^3$ $\mu$m), and is still smaller than the typical ballistic length 50–100 $\mu$m. Distinct solitons will start to appear right after the shock. Observation of the electric charge carried by a distinct soliton will provide a direct measurement of fractional charge. The full decay of this packet is about $10^3$ times longer. These estimates suggest that nonlinear effects can be observed in a nanosecond range.

We have benefited from discussions with O. Agam, I. Krichever, B. Spivak, A. Zabrodin, V. Goldman, and W. Kang. Our special thanks to I. Gruzberg, D. Gutman, R. Teodorescu. P. W. and E. B. were supported by NSF MRSEC No. DMR-0213745 and NSF No. DMR-0220198. A. G. A. was supported by NSF No. DMR-0348358.

*Also at Landau Institute of Theoretical Physics, Moscow, Russia.


[1] L. Landau, J. Phys. U.S.S.R **5**, 71 (1941).
[2] M. Schick, Phys. Rev. **166**, 404 (1968); F. D. M. Haldane, J. Phys. C **14**, 2585 (1981).
[3] A. Jevicki and B. Sakita, Nucl. Phys. **B165**, 511 (1980).
[4] Z. Ha, *Quantum Many-Body Systems in one Dimension* (World Scientific, Singapore, 1996).
[5] X. G. Wen, Phys. Rev. B **41**, 12 838 (1990).
[6] A. G. Abanov and P. B. Wiegmann, Phys. Rev. Lett. **95**, 076402 (2005).
[7] A. Jevicki, Nucl. Phys. **B376**, 75 (1992).
[8] H. Awata *et al.*, Phys. Lett. B **347**, 49 (1995).
[9] I. Andrić, A. Jevicki, and H. Levine, Nucl. Phys. **B215**, 307 (1983); I. Andrić and V. Bardek, J. Phys. A **21**, 2847 (1988).
[10] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Butterworth-Heinemann, Oxford, 1987), Vol. 101.
[11] *Singular Limits of Dispersive Waves*, edited by N. M. Ercolani *et al.* (Plenum, New York, 1994).
[12] v is a Hermitian operator with respect to the inner product $\langle f, g\rangle = \oint dx_i|\Delta(x)|^{2\lambda}f(x)g(x)$, where $\Delta^\lambda(x_i) = \prod_{i\neq j}\{\sin[\frac{\pi}{L}(x_i - x_j)]\}^\lambda$ is the ground state of (3).
[13] Y. Matsuno, *Bilinear Transformation Method* (Academic, Orlando, 1984).
[14] J. Satsuma and Y. Ishimori, J. Phys. Soc. Jpn. **46**, 681 (1979).
[15] S. Y. Dobrokhotov and I. M. Krichever, Math. Notes Acad. Sci. USSR **49**, 583 (1991).
[16] E. Bettelheim, A. G. Abanov, and P. Wiegmann, cond-mat/0606778.
[17] A. P. Polychronakos, Phys. Rev. Lett. **74**, 5153 (1995).
[18] J. B. Whitham, *Linear and Nonlinear Waves* (Wiley-Interscience, New York, 1974).
[19] A. V. Gurevich and L. P. Pitaevskiĭ, Sov. Phys. JETP **38**, 291 (1974).
[20] Y. Matsuno, Phys. Rev. E **58**, 7934 (1998).
[21] W. Kang (private communication).
[22] I. J. Maasilta and V. J. Goldman, Phys. Rev. B **57**, R4273 (1998).