P H Y S I C A L   R E V I E W   L E T T E R S

Generic Quantum Ratchet Accelerator with Full Classical Chaos

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A simple model of quantum ratchet transport that can generate unbounded linear acceleration of the quantum ratchet current is proposed, with the underlying classical dynamics fully chaotic. The results demonstrate that generic quantum ratchet transport can occur with any type of classical phase space structure. The quantum ratchet transport with full classical chaos is also shown to be very robust to noise due to the large linear acceleration afforded by the quantum dynamics. One possible experiment allowing observation of these predictions is suggested.

Understanding and exploring aspects of quantum transport is of great importance in a variety of contexts, including, for example, nanoscale electronic devices [1], atom optics with optical lattices [2], and coherently controlled photocurrents in semiconductors [3]. Of particular interest is directed transport without a biased field [4,5], an important phenomenon often called ratchet transport. While ultimately both quantum and classical ratchet transport are of similar origin, namely, breaking of certain spatio-temporal symmetries, it is challenging to understand their relationship in general [6]. Indeed, as suggested by quantum tunneling induced current reversal [7–10], knowledge of classical transport should be applied with caution in understanding quantum transport.

The quantum-classical contrast in ratchet transport is all the more complex and intriguing in chaotic systems, where the symmetry breaking that underlies directed quantum transport is necessarily entangled with the many aspects of quantum chaos, e.g., dynamical localization, quantum resonance, and quantum anomalous diffusion. As such, chaotic model systems are especially attractive for studies of quantum ratchet transport, both theoretically and experimentally [10–16]. Consider the familiar kicked-rotor model and its extensions as examples. It has been shown that dynamical localization can saturate quantum directed current [10], and that quantum resonance can induce linear acceleration of the directed current without saturation, even when the underlying classical dynamics is fully chaotic [15,16]. The latter result is quite surprising, insofar as the ensemble-averaged classical acceleration rate of directed current should, according to a recent classical theorem [11], vanish when the classical dynamics is completely chaotic. However, this type of directed quantum transport occurs only for isolated values of the effective Planck constant and is extremely vulnerable to noise [15], thereby making its experimental observation challenging.

In this Letter we propose a novel quantum ratchet transport model, called a ''quantum ratchet accelerator,'' that

also displays unbounded linear acceleration of the quantum ratchet current while the underlying classical dynamics is fully chaotic. However, this quantum ratchet accelerator is generic: it is unrelated to quantum resonance, and the acceleration rate is in general nonzero for an arbitrary effective Planck constant. This firmly establishes, for the first time, that generic quantum ratchet transport is possible with full chaos in the underlying classical dynamics, and hence does not require mixed classical phase space structures. Equally important, quantum ratchet transport in our quantum ratchet accelerator is robust to noise effects. As shown below, noise of considerable intensity will saturate the otherwise accelerating ratchet current, but large ratchet currents that are absent in the classical system are still observed. One possible experimental scenario to observe the proposed acceleration is also suggested below.

The model proposed below is an extension of the kicked Harper model, a paradigm of quantum chaos that may not display dynamical localization [17–19]. The kicked Harper model has attracted enormous interest and is closely related to driven harmonic oscillator systems [20], kicked charges in a magnetic field [21], and driven electrons on the Fermi surface [22]. Here we reapproach this paradigm from the perspective of quantum ratchet transport, by breaking the spatial symmetry of the kicking potential. Specifically, we consider the following ''delta-kicked'' model,

$$H = L\cos(p) + KV(q)\sum_n \delta(t - n); \qquad (1)$$

$$V(q) = \cos(q + \phi_1) + \eta\sin(2q + \phi_2). \qquad (2)$$

Here all variables should be understood as appropriately scaled and hence dimensionless. In particular, q and p are position and momentum variables, n is an integer, and L, K, ϕ_1, ϕ_2, η are system parameters. The commutation relation between q and p gives the effective Planck constant $\hbar$, i.e., $i\hbar = [q, p]$. The model reduces to the original kicked Harper model for $\phi_1 = 0$ and $\eta = 0$, and hence can

be regarded as a generalized kicked Harper model. For future theory development we note that Dana [23] has studied some basic properties of the quasienergy bands associated with any function $V(q)$ that has a period of 2π.

In presenting detailed results we focus mainly on one typical case, where $\phi_1 = \phi_2 = 0$, $\eta = 1.0$, and $K = 2L = 3.0$. Related cases will also be emphasized when the comparison becomes enlightening. The phase space structure for the associated classical map in a unit cell is shown in Fig. 1(a), where no stable islands are found on a very fine scale. Hence, for all practical purposes, the classical dynamics is fully chaotic. Indeed, the broken spatial symmetry, as evident in the kicking potential, is invisible in Fig. 1(a). By contrast, for smaller values of K and L, the phase space becomes structured; a mixture of chaotic and integrable motions for some parameters [Fig. 1(b)], or predominantly integrable [Fig. 1(c)]. Concerning the results in Figs. 1(b) and 1(c), we also note (i) that the phase space structure is clearly not symmetric if $q \rightarrow -q$, and (ii) that some phase space invariant curves are extended in momentum space, hence allowing unbounded acceleration of classical trajectories. The intuitive character of this acceleration, coupled to the slowness of the q-averaged acceleration rate of the directed currents (described below), make it far less interesting than the chaotic case emphasized in this Letter.

Consider now the quantum current $\langle p \rangle$ for an initial quantum state $|0\rangle$ with $p|0\rangle = 0$. Note that this initial state is symmetric in time and space, and the corresponding classical state should be described by a zero-momentum ensemble with q uniformly distributed from 0 to 2π. To emphasize that our results are unrelated to quantum resonance, consider an effective Planck constant $\hbar = 2\pi/(6 + \sigma_g)$, with $\sigma_g = (\sqrt{5} - 1)/2$. This ensures that $\hbar/2\pi$ is as irrational as possible. As seen from Fig. 2, the computed quantum current (solid line) displays beautiful linear acceleration without saturation. Indeed, the absence of dynamical localization in this system implies that the acceleration should continue, unsaturated, with additional kicks. The inset shows the highly asymmetric probability density distribution after 1000 kicks, in terms of the basis states $|m\rangle$, with $p|m\rangle = m\hbar|m\rangle$. Note that the acceleration rate, defined as the average increase of quantum ratchet current after each kick, is as large as 0.42, yielding a quantum current that can be orders of magnitude larger than in previous studies [10,12,14]. By contrast, an inspection of the classical transport behavior (dotted line) shows that there is no systematic acceleration in the classical current. Rather, the classical current quickly saturates and remains extremely small at all times. Figure 2 also shows that the acceleration rate of the quantum current is much smaller, but still nonzero, for smaller values of K and L that give rise to either mixed or predominantly integrable dynamics [see Figs. 1(b) and 1(c)]. This makes it clear that all types of classical phase space structures, i.e., fully chaotic, mixed, or completely integrable, can be exploited in constructing a quantum ratchet accelerator, thereby adding an important feature to current knowledge regarding quantum-classical correspondence in ratchet transport [6,10–12,14,19]. The full chaos case is our focus here because it is counterintuitive and can give the largest ratchet current. Because the quantum dynamics of one-
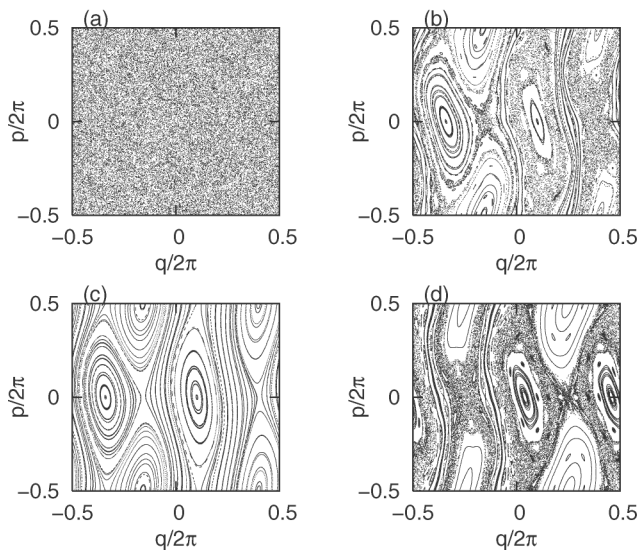

FIG. 1. Classical phase space structures for the generalized kicked Harper models with Hamiltonian given by Eq. (2). In (a)–(c) the kicking potential is given by $V(q) = \cos(q) + \sin(2q)$, and in (d) $V(q) = \sin(q) + \cos(2q)$. $K = 2L = 3.0$ in (a), $K = 2L = 1.0$ in (b) and (d), and $K = 2L = 0.4$ in (c). As shown in Fig. 2, cases (a)–(c) can be exploited to generate quantum ratchet current. Not so for case (d) due to a spatiotemporal symmetry.
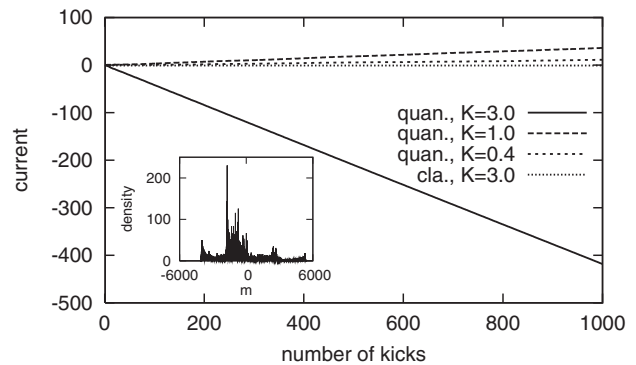

FIG. 2. Unbounded linear acceleration of the quantum ratchet current (solid line) with full classical chaos shown in Fig. 1(a). The classical ratchet current (dotted line) shows no linear acceleration, with its value indistinguishable from zero on this scale. The quantum current associated with mixed ($K = 2L = 1.0$) or predominantly integrable classical dynamics ($K = 2L = 0.4$) is also shown for comparison. The inset shows the highly asymmetric probability density distribution of the quantum state after 1000 kicks in representation of momentum eigenstates, for the fully chaotic case.

dimensional kicked systems can often be mapped onto that of many-body lattice systems [24], results here imply that quantum many-body systems in the thermodynamic limit can generically display directed transport, even when the associated classical dynamics is fully chaotic.

The sharp contrast between quantum and classical dynamics can be qualitatively understood as follows: classically, a system with fully developed chaos will quickly forget its history. Hence, any classical mechanism related to symmetry breaking can operate only within the relaxation time scale, and the ensemble-averaged classical current should quickly saturate, leading to a vanishing acceleration rate of the directed current. However, while all classical phase space invariant curves get broken in the case of full chaos, their remnants, typically cantorus-like structures that are much smaller than $\hbar$, can still play a key role in the quantum dynamics. In particular, the cantorus-like structures present a strong barrier for a time-evolving quantum wave packet to pass through, and can even attract concentrations of quantum states [17]. Because these remnants are just as asymmetric as the classical invariant curves seen in Figs. 1(b) and 1(c), the broken symmetry can be clearly manifested in the quantum dynamics, giving rise to an unbounded linear acceleration of the quantum current. Confirming this understanding, we note that if no classical invariant curves are extended in the momentum space (e.g., when $2K = L$), then we found no linear acceleration of the quantum current for any type of classical phase space structure.

It should also be noted that broken symmetry in the q space alone, as indicated on one stroboscopic plot, does not suffice for ratchet transport. Consider, for example, a kicking potential $V(q)$ with $\phi_1 = -\pi/2$, $\phi_2 = \pi/2$, and $\eta = 1$, i.e., $V(q) = \sin(q) + \cos(2q)$. In this case, both classical and quantum currents are zero. Indeed, one can easily prove the existence of a hidden symmetry: the dynamics are invariant to $q \to \pi - q$ and $p \to -p$. Such a spatio-temporal symmetry can be clearly seen if we examine either the left or right half of the phase space cell shown in Fig. 1(d). Averaging over q then causes loss of directed current.

To shed more light on the quantum ratchet accelerator, consider now the $\hbar$ dependence of the associated acceleration rate, for a sampling regime $0.9 \leq \hbar \leq 1.2$ that includes the $\hbar = 2\pi/(6 + \sigma_g)$ case examined in Fig. 2. Figure 3 displays the results for 40 values of $\hbar$ ($\hbar = 0.9 + j/100$, $j = 0 - 30$, and $\hbar = 1.0 + m/1000$, $m = 1 - 9$). The behavior of the acceleration rate seen in Fig. 3 is highly nontrivial, and the quantum current is seen to reverse its sign as $\hbar$ increases from 1.03 to 1.04. This current reversal, unrelated to quantum tunneling, reflects a redistribution of the concentration of quantum amplitudes on the remnants of classical invariant curves as $\hbar$ varies. Further, it suggests that the direction of quantum ratchet current may be controlled by actively tuning the effective
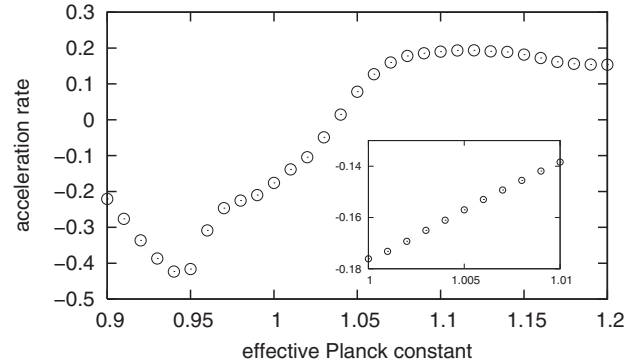

FIG. 3. The $\hbar$-dependence ($0.9 \leq \hbar \leq 1.2$) of the linear acceleration rate of the quantum ratchet current, in a case of full classical chaos. The system parameters are the same as in Fig. 1(a). The smooth $\hbar$ dependence on a smaller scale is shown in the inset.

Planck constant. Finally, note that the inset of Fig. 3 demonstrates that the $\hbar$ dependence is smooth, suggesting the possibility of a quantitative theory of quantum ratchet transport. Such a quantum theory for fully chaotic systems, far beyond the scope of this work, would offer a new tool in understanding quantum chaos.

Because the exposed quantum ratchet transport with full classical chaos is based entirely on nonclassical effects, one might expect it to be fragile when subject to noise. This is not the case. As one key advantage of a generic quantum ratchet accelerator, the quantum ratchet transport has built-in capabilities to fight against detrimental noise effects. That is, noise effects do not easily destroy the ratchet transport, thanks to the very large acceleration mechanism inherent in the ratchet. To see this, consider one amplitude-noise model as well as one phase-noise model (these models do not quantitatively represent decoherence effects if the system is coupled to a real bath). In the first model, we assume that the kicking field strength K is scaled by a randomly fluctuating term $[1 + A(\xi - 0.5)]$, where ξ is a random variable uniformly distributed in [0, 1]. In the second model, we expand the evolving quantum state in terms of momentum eigenstates, and then introduce, after each kick, random quantum phases $\exp[iB2\pi\xi]$ to each individual momentum eigenstate to dephase them. The results are shown in Fig. 4. Although in both models higher noise intensity, characterized by larger A or B, suppresses the linear acceleration of quantum ratchet current to an increasing degree, substantial ratchet currents still survive in the presence of noise. Remarkably, even when the fluctuation in K reaches 10% of its average value, or when the random phase fluctuation periodically introduced to momentum basis states is $\approx 0.1\pi$, the quantum ratchet currents, although saturated, still remain orders of magnitude larger than the classical currents and the quantum currents obtained in, e.g., fully chaotic delta-kicked rotor systems [12,14].
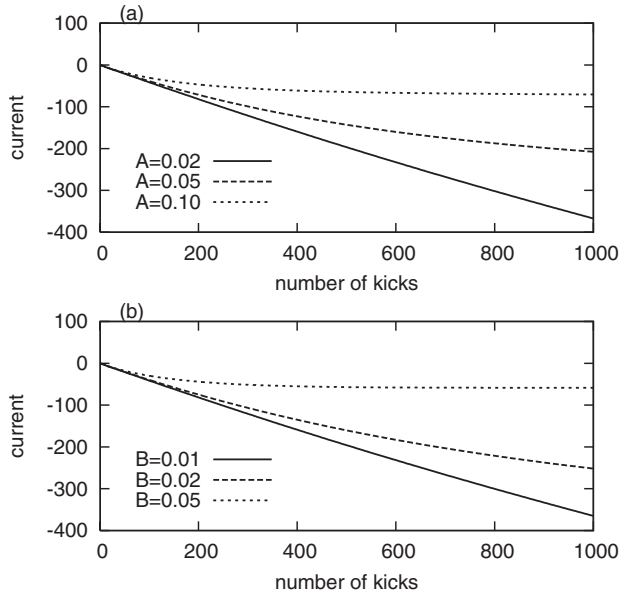
FIG. 4.   The robustness of quantum ratchet transport against noise (see the text for two noise models) in a quantum ratchet accelerator with full classical chaos shown in Fig. 1(a). The noiseless case is shown as the solid line in Fig. 2. Effects of amplitude noise are shown in (a), and effects of phase noise are shown in (b). In each case 1000 realizations of noise history are used to obtain the average behavior. A and B represent the noise intensity. For $A = 0.10$ or $B = 0.05$, the current is seen to saturate within 1000 kicks. For the cases with smaller A or B, saturation will occur much later.

Neither the kicked Harper model nor its generalizations studied here have been experimentally realized with atom optics. Consider then an alternative and interesting possibility: according to a general result of Dana [21], the time evolution operator of our model can be exactly mapped to that of a subensemble of kicked charges in a magnetic field. The Hamiltonian of the latter is given by $H = \Pi^2/2 + \lambda U(x, t)\sum_s \delta(t - s/4)$, with Π the kinetic momentum of the kicked charge in a magnetic field. The kicking potential $U(x, t)$ should satisfy, for a particular value of x_0 associated with a constant of the motion that defines the subensemble, $U(x_0 - x, t) \propto [L \cos(x)f(t) + KV(-x)f(t - 1/4) + L \cos(x)f(t - 1/2) + KV(x)f(t - 3/4)]$, where $f(t)$ satisfies $f(t) = f(t + 1)$, $f(0) = 1$, and $f(1/4) = f(1/2) = f(3/4) = 0$. This type of kicking potential should be achievable with modern pulse shaping techniques. Quantum ratchet transport with full classical chaos would then be observed in the net charge transport in coordinate space, or in the asymmetric kinetic momentum distribution of the kicked charges.

In conclusion, we have proposed a generic Hamiltonian model of quantum ratchet transport where the underlying classical dynamics is fully chaotic. Our results show that generic quantum ratchet transport can occur with any type of classical phase space structure. Further, the exposed $\hbar$ dependence of the acceleration rate of the quantum ratchet current represents a novel and challenging issue in quantum chaos. Finally, the robustness of the quantum ratchet transport against noise is also demonstrated. Future work will also consider active manipulation of the quantum ratchet transport, by taking advantage of the $\hbar$ dependence of the acceleration rate and by seeking an optimized kicking potential to generate ratchet transport most efficiently.

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