Anomalously Long Passage through a Rounded-Off-Step Potential due to a New Mechanism of Multidimensional Tunneling

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The fully complex domain semiclassical theory based upon the complexified stable-unstable manifold theory, which we have developed in our recent studies, is successfully applied to explain anomalous tunneling phenomena numerically observed in a periodically modulated round-off-step potential. Numerical experiments show that tunneling through the oscillating step potential is characterized by a spatially nondecaying tunneling tail and an anomalously slow relaxation. The key is the existence of a critical trajectory exhibiting singular behavior, and the analysis of neighboring trajectories around it reproduces the essence of such anomalous phenomena.

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The tunneling effect is one of the most fundamental processes, and most of the tunneling processes are contributed to by multiple degrees of freedom. However, there is no satisfactory theory for understanding and interpreting the underlying mechanism of multidimensional tunneling processes in terms of classical trajectory concepts. As yet, we have no essentially new theoretical framework of tunneling phenomena alternative to the instanton theory of one-dimensional barrier penetration [1,2]. The essential difficulty of multidimensional tunneling is due to the presence of chaos, which makes it very hard to tackle the tunneling phenomena by means of the classical trajectory concepts. On the other hand, the presence of chaos remarkably influences the nature of tunneling phenomena [3–5], which has actually been observed in experiments [6–8].

In order to investigate the underlying classical mechanism of tunneling, it is natural to use the semiclassical theory extended into the complex domain [1,2]. When considering tunneling in classically nonintegrable systems, the nature of complex trajectories contributing to tunneling is remarkably affected by the presence of complex domain chaos: the entanglement of complexified stable and unstable manifolds [5,9]. Therefore, it is wise to start with the study of simplified models which intrinsically involve the essence of multidimensionality in tunneling. Applying the complex semiclassical theory to simple model systems, our group has investigated in a series of recent works the underlying mechanism of multidimensional tunneling affected by classical chaos [9–11] and has proposed a new universal tunneling mechanism based on geometric and dynamical properties of the complexified stable and unstable manifolds, W_s and W_u.

In this Letter, we demonstrate the utility of our complex semiclassical theory by applying it to a new class of more-than-one-dimensional tunneling system, and we show that the complexified stable-unstable manifold theory predicts a novel tunneling phenomenon and, further, that its anomalous properties are described within the framework of our semiclassical theory.

When applying the complex semiclassical method to the tunneling problem, we have to prepare a set of initial points of classical trajectories as the classical counterpart of the quantum initial (or boundary) condition, which forms a hypersurface $\mathcal{I}$ in the complex phase space. When chaotic tunneling emerges, the surface $\mathcal{I}$ always intersects with a stable manifold W_s in the complex domain but not in the real domain. The intersections form isolated points, which we call *critical points*. Trajectories contributing to tunneling that start from a set in $\mathcal{I}$ located close to a critical point are guided by the complexified stable manifold W_s and approach the real phase space. They are finally scattered along the unstable manifold W_u and end at a manifold where observation is done. As a result, the tunneling probability affected by the nature of the stable and unstable manifolds is observed. This new mechanism is essentially different from the instanton mechanism [2]. For example, the characteristic tunneling phenomena such as fringed tunneling [10] and the plateau energy spectrum [11] peculiar to multidimensional barrier systems are manifestations of the new tunneling mechanism.

The model system we consider in this Letter is a round-off-step potential perturbed by a periodic force

$$H(Q, P, \omega t) = \tfrac{1}{2}P^2 + (1 + \epsilon \sin\omega t)[1 + \exp Q]^{-1}. \quad (1)$$

To consider the nature of the step potential, we first take the round-off box potential $V_B(Q, \omega t) = (1 + \epsilon \sin\omega t) \times ([1 + \exp(-(Q - Q_B))]^{-1} - [1 + \exp(-Q)]^{-1})$ instead of the step potential. The box potential V_B has an unstable periodic orbit at the top of barrier accompanied by stable and unstable manifolds W_s and W_u. The step potential is obtained by taking the limit $Q_B \to -\infty$, and, hence, the unstable periodic orbit goes to $-\infty$, but there exists a critical invariant manifold that plays the same role of the stable manifold W_s: It separates the input particles with any initial energies into transmissive ones and reflective ones. Thus, we, hereafter, call that surface the stable manifold W_s.

Suppose that an incident plane wave with a constant energy $E_1(= P_1^2/2)$ comes from $Q = +\infty$. If E_1 is taken small enough, then the quantum probability observed in the transmissive side ($Q < 0$) is caused by tunneling.

Then no intersections between W_s and $\mathcal{I}$ appear in the real domain. Nevertheless, there always exist intersections between W_s and $\mathcal{I}$ in the complex domain, which form isolated points t_{1c}'s on the initial complex time plane t_1 at $Q = Q_1(\gg 1)$ [10]. t_{1c} is the critical point, and the trajectory leaving from t_{1c} is called the critical trajectory.

The existence of the intersections can be proven by using the Melnikov method applied to classical solutions on W_s [10], and the imaginary depth of t_{1c} is estimated by

$$\mathrm{Im}\,(\omega t_{1c}) = \cosh^{-1}((1 - E_1)/\epsilon A), \qquad (2)$$

where A is a constant obtained from calculation of the Melnikov integral and depends on the shape of the potential and ω.

Therefore, it is expected that there are tunneling phenomena subject to the new tunneling mechanism. Figure 1 shows the fully quantum mechanical scattering eigenstates as functions of Q_2 for various values of perturbation strength $\epsilon = 0, 0.05, 0.1, 0.2$. At $\epsilon = 0$, i.e., the unperturbed case, the tunneling tail penetrating into the potential wall drops off very quickly, which can be explained by the instanton. However, once the perturbation is applied, tunneling rates are extremely enhanced, and the probability amplitude of each eigenstate does not drop and keeps nearly a constant value over the transmissive region ($Q < 0$), which increases with ϵ. The physical origin of such remarkable effects should be explained by the new tunneling mechanism.

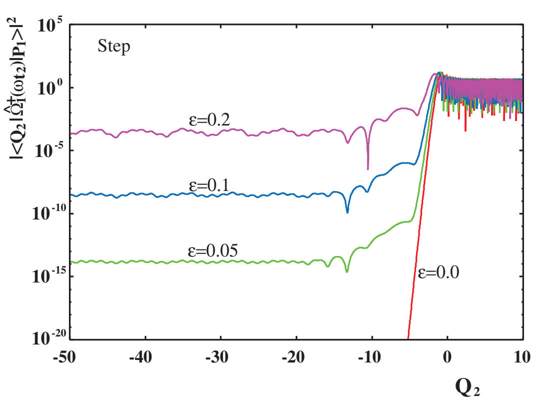

FIG. 1 (color online). Probabilistic amplitudes of scattering eigenstates for the rounded-off-step potential at four representative values of ϵ. The parameters are chosen as $E_1 = 0.75$, $\omega = 0.3$, and $\hbar = 1000/(3\pi \times 2^{10}) \sim 0.1036$.

To do this, we have to estimate semiclassical weights of trajectories which are guided by the complexified W_s. The semiclassical weight is, roughly speaking, determined by the imaginary part of the classical action of contributing complex trajectories as $\sim \exp\{-2\,\mathrm{Im}S_\Omega/\hbar\}$, where the classical action S_Ω is defined by

$$S_\Omega(Q_2, t_2, Q_1, E_1) \equiv \int_{Q_1}^{Q_2} P dQ - \int_{t_1}^{t_2} H(Q, P, \omega t) dt + E_1(t_2 - t_1), \qquad (3)$$

where subscripts 1 and 2 of dynamical variables indicate input and output, respectively. From consideration of the boundary condition for complex trajectories, we see that the coordinate Q_1 and energy $E_1(= P_1^2/2)$ at the input side are quantities observed and should be taken as real values, and Q_2 and t_2 at the end should also be real. On the other hand, the initial time t_1 is unobserved and allowed to be a *complex* variable.

In order to obtain complex trajectories going to the tunneling side, their integration paths should be chosen to have an appropriate topology with respect to singularities in the complex time plane [10,11]. The integration path chosen consists of a piecewise straight contour $\mathcal{C}$: $t_1(\in \mathbf{C}) \to t_0(\in \mathbf{R}) + i\,\mathrm{Im}t_1 \to t_0 \to t_2(\in \mathbf{R})$. Since the complex trajectories subject to the new tunneling mechanism have initial times t_1's in the close neighborhood of t_{1c}, then they evolve along $\mathcal{C}$ following the complexified W_s and pass over the potential edge just before $t = t_0 + i\,\mathrm{Im}t_1$. After that, $\mathcal{C}$ is bent at $t = t_0 + i\,\mathrm{Im}t_1$, goes down to the real axis, and is bent again at $t = t_0$. Until reaching t_0, the trajectories undergo complex time evolution, but in the region ($t_0 < t < t_2$) they take almost real values in Q and P and penetrate into the oscillating flattop of potential, finally going toward $Q = -\infty$ with a very small $|P| \ll 1$ along the real W_s.

Thus, the tunneling particle does not significantly gain the imaginary action $\mathrm{Im}S_\Omega$ along the path $t_0 \to t_2$. Most of $\mathrm{Im}S_\Omega$ is gained in the complex time evolution, along the path $t_1 \to t_0 + i\,\mathrm{Im}t_1 \to t_0$. As a result, $\mathrm{Im}S_\Omega$ of all of the contributing trajectories take almost the same value, which can be well approximated by that of the critical trajectory launched at t_{1c}. The tunneling amplitude $\sim \exp\{-2\,\mathrm{Im}S_\Omega/\hbar\}$ thus becomes nearly constant in the tunneling region and is independent of the position of the detector at $Q = Q_2(<0)$. This is the reason the amplitude of the eigenstate becomes nearly constant over the tunneling regime. $\mathrm{Im}S_\Omega$ along the critical trajectory can be evaluated by applying the Melnikov method to the computation of the action integral of Eq. (3) [11]. Table I shows the tunneling rates estimated by the Melnikov method, which are compared with those of the fully quantum calculation. Our theoretical predictions are in acceptable agreement with the quantum results.

The tunneling rate increases with ϵ. The key to understand this is the imaginary depth of t_{1c}, which is estimated

TABLE I. Tunneling rate.

ϵ	$\exp\{-2\,\mathrm{Im}S_\Omega/\hbar\}$	Numerical
0.2	$0.596\,391 \times 10^{-3}$	$O(10^{-4})$
0.1	$0.138\,531 \times 10^{-7}$ exp	$O(10^{-9})$
0.05	$0.226\,217 \times 10^{-12}$	$O(10^{-14})$

by Eq. (2). Since t_{1c}, with an increase of ϵ, goes toward the real axis from the deeper imaginary side, then the trajectory undergoes a shorter complex time evolution, and $\mathrm{Im}S_\Omega$ becomes smaller.

From the above discussion, it can be concluded that the remarkable enhancement in the tunneling rate by the perturbation is accounted for by the new tunneling mechanism connected with the geometrical structure of W_s.

We can further confirm that application of the fully semiclassical method to the model system completely reproduces the tunneling part of the scattering eigenstate. The scattering eigenstate is calculated with the wave operator, and its semiclassical expression is given by [12]

$$\langle Q_2|\hat{\Omega}_1^+(t_2)|P_1\rangle \sim \sum_{\mathrm{c.t.}} \lim_{Q_1\to\infty} \sqrt{\frac{|P_1|}{2\pi\hbar M}} e^{iP_1Q_1/\hbar}\left[\frac{M}{P_1}\frac{\partial^2 S_\Omega}{\partial E_1 \partial Q_2}\right]^{1/2} \times \exp\left\{\frac{i}{\hbar}S_\Omega(Q_2,t_2,Q_1,E_1)\right\}. \quad (4)$$

The notation $\sum_{\mathrm{c.t.}}$ means summing over all of the contributing (complex) classical trajectories satisfying the boundary condition: Both Q_1 and E_1 are real at the input side and real Q_2 and real t_2 at the output. Then t_1 is regarded as a complex search parameter to find the trajectory satisfying the boundary condition of the semiclassical operator. Now we introduce the set of initial times $t_1 \in \mathbf{C}$ of the contributing trajectories: $\mathcal{M} = \{t_1|\mathrm{Im}Q(t_2 - t_1, t_1, P_1, Q_1) = 0\}$ [9,12].

Figure 2(a) shows the $\mathcal{M}$ set at $\epsilon = 0.2$ together with the critical points t_{1c}'s marked by an $\times$, i.e., the intersection of the complex t_1 plane with W_s. The critical points have the same imaginary part and are aligned at the real period $T = 2\pi/\omega$ of the perturbation. The $\mathcal{M}$ set forms a set of continuous curves, namely, complex branches. The branches labeled 1–4 make a major contribution to the tunneling wave. Each major contributing branch passes close to the corresponding t_{1c}. The appearance of t_{1c} accompanied by the characteristic branch is the landmark of the chaotic tunneling for which the tunneling particle is guided by W_s.

Figure 2(b) shows the scattering eigenstate semiclassically reproduced together with the quantum result. The agreement is very good. To obtain a well-converged semiclassical eigenfunction, complex branches over 300 periods have to be included in the calculation. Figure 2(c) shows the contributions of the first 10 branches (i.e., 10 periods). Contributions of the branches relax very slowly with an increase of the branch number, which is the manifestation

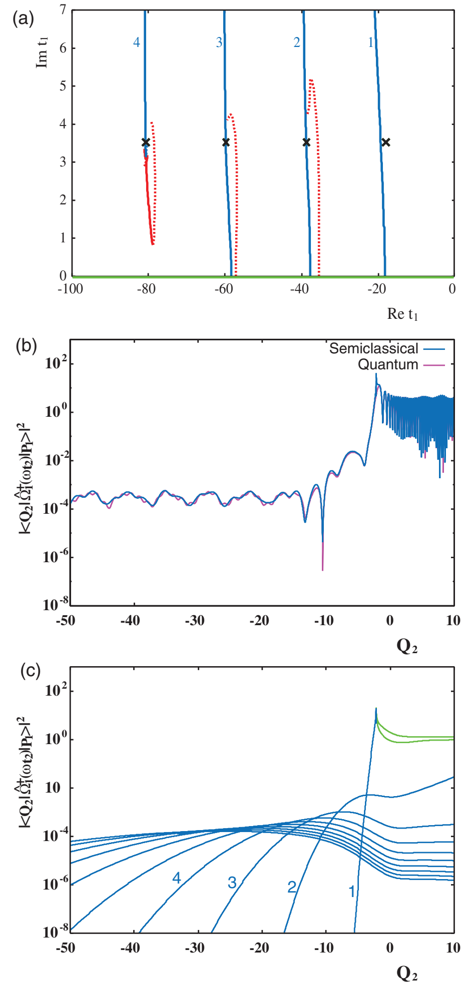

FIG. 2 (color online). Semiclassical results at $\epsilon = 0.2$. (a) The $\mathcal{M}$ set. (b) The semiclassical scattering eigenstate is compared with the quantum result. (c) Contributions to the scattering eigenstate from the first 10 branches of the $\mathcal{M}$ set depicted in (a).

of the slow convergence of the scattering eigenstate. Such slowly varying behavior comes from the fact that the momenta $|P|$'s of the trajectories contributing to each branch become extremely small when they go over the oscillating flattop of the potential.

Actually, the momenta $|P|$'s of trajectories starting at a neighborhood of t_{1c} are estimated by $|P|^2 \sim O(\epsilon\omega) \times |t_1 - t_{1c}|$. Such a very slow passage is just the character of the contributing trajectories guided by the critical trajectory.

Notice that the tunneling part of the scattering eigenfunction is constructed by superposing the transmitted components of sequential pulse wave input at $Q = \infty$ [12] continuously in time. The very slow convergence of the scattering eigenfunction implies that each pulse wave transmitted past the barrier is accompanied with a very slowly decaying tail. Such an anomalous behavior will be directly observed by a single pulse tunneling. We show fully the quantum transient behavior of tunneling by inputting a spatially localized wave packet at $Q \gg 1$.

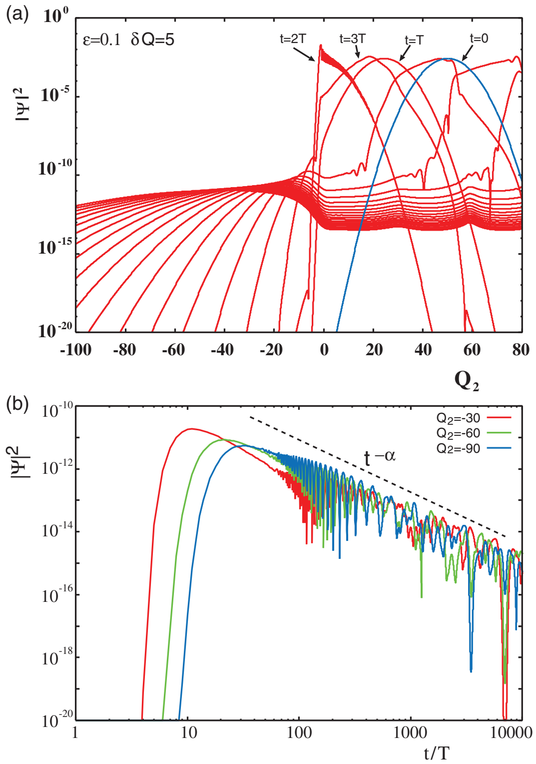

FIG. 3 (color online). Time evolution of an initially localized wave packet Ψ scattered by the oscillating step potential at $\epsilon = 0.1$. The width of the initial packet is given as $\delta Q = 5.0$ and the average energy as $\langle E \rangle = 0.75$. (a) The time evolution for the first 20 periods. (b) Change of the tunneling probability at various points $Q_2 = -30, -60, -90$.

Let us see numerical results: Figure 3(a) shows the time evolution of an initially localized wave packet Ψ for the first 20 periods scattered by the step. The wave packet penetrating into the potential wall spreads widely over the classically forbidden region as the time elapses, and it finally, at $t = 20T$ ($T = 2\pi/\omega$), becomes distributed almost flatly over the tunneling region forming a similar distribution as the scattering eigenstate with the same tunneling probability [see the data of $\epsilon = 0.1$ in Fig. 1]. Figure 3(b) shows the change of tunneling probability for a long term evolution observed at various positions of the Q_2's. The lapse time t_m spent for the peak of tunneling wave reaching the detector at Q_2 becomes nearly proportional to its distance from the potential entrance at the origin $t_m \propto |Q_2|$. On the other hand, the probabilities observed at the fixed detectors are accompanied by a long time tail characterized by a power-law decay $\propto t^{-\alpha}$, with $\alpha \sim 1.7$, which means that the shape of the initial packet accompanied by an exponentially decaying tail is anomalously distorted. In spite of the fact that the motion of the particle far from the step (i.e., $Q \ll -1$) is free, the whole process of the tunneling penetration is anomalously slow. This is, of course, a reflection of the anomalously slow complexified classical motion around the critical trajectory. A complete semiclassical description of the anomalously slow relaxation is possible, for which estimation of weights of trajectories starting at a neighborhood of $t_{1c}(\propto |t_1 - t_{1c}|^{\beta}, \beta \sim 1)$ is required [11], but it is beyond the scope of this Letter, and it will be published elsewhere [13].

The appearance of this type of tunneling seems to be common for a certain class of potentials having a wide flattop: Actually, similar tunneling phenomena are observed for the round-off box potential $V_B(Q, \omega t)$.

Finally, we conclude that, when a certain class of potentials with a wide flattop, e.g., rounded-off-step potential (and box potential), is perturbed by a periodic force, the tunneling rates of the scattering eigenstate and of the collided wave packet are remarkably enhanced by the periodic force and are controlled by the strength of the force. For the scattering eigenstate, the underlying classical mechanism is well understood with the complex semiclassical theory combined with the stable-unstable manifold theory. The complex semiclassical theory provides a classical trajectory picture of multidimensional tunneling, and it will be very useful as a new methodology for designing and controlling complicated tunneling phenomena in multidimensional systems.

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