Nonequilibrium Quantum Criticality in Open Electronic Systems

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(Received 10 July 2006; published 8 December 2006)

A theory is presented of quantum criticality in open (coupled to reservoirs) itinerant-electron magnets, with nonequilibrium drive provided by current flow across the system. Both departures from equilibrium at conventional (equilibrium) quantum critical points and the physics of phase transitions induced by the nonequilibrium drive are treated. Nonequilibrium-induced phase transitions are found to have the same leading critical behavior as conventional thermal phase transitions.

DOI: 10.1103/PhysRevLett.97.236808 PACS numbers: 75.10.Jm, 05.30.−d, 71.10.−w

A central issue in condensed matter physics is the behavior of systems as one tunes parameters (for example, pressure or magnetic field) so as to change the symmetries characterizing the ground state [1–4]. The parameter values at which the ground state symmetries change (for example, from ferromagnetic metal to paramagnetic metal) define a quantum phase transition point (quantum critical point). At quantum phase transitions, spatial and temporal fluctuations are coupled, so that continuous quantum phase transitions in equilibrium systems are typically described by critical theories involving an effective dimensionality $d_{\rm eff}$ greater than the spatial dimensionality d.

While equilibrium quantum phase transitions have been extensively studied, the generalization to nonequilibrium conditions raises a largely open class of questions. Nonlinear transport near a superconductor-insulator phase transition [5,6] and a ferromagnetic transition driven by current flow in a closed one-dimensional system [7] have been studied; however, a general systematic understanding is lacking.

In this Letter we formulate a theory of nonequilibrium quantum criticality in itinerant-electron systems coupled to reservoirs (cf. upper panel of Fig. 1) with which particles may be exchanged. Nonequilibrium is imposed by differences between reservoirs; our systems are therefore subject to a time-independent drive and are not characterized by any conserved quantities. A generic phase diagram is shown in the lower panel of Fig. 1: we take a system which at temperature $T = 0$ may be tuned through an equilibrium quantum critical point by varying a parameter δ through a critical value δ_c and determine the changes induced by a nonequilibrium drive (generically denoted as V). Of particular interest is the transition generated by V if the $V = 0$ system is ordered (vertical arrow in lower panel of Fig. 1).

Analysis of nonequilibrium systems proceeds from the time dependent density matrix $\hat{\rho}(t)$ defined in terms of a Hamiltonian $\hat{H}$ and an initial condition $\hat{\rho}(t_{\rm init})$ via $\hat{\rho}(t) = e^{-i\hat{H}(t-t_{\rm init})}\hat{\rho}(t_{\rm init})e^{i\hat{H}(t-t_{\rm init})}$. The open systems we consider possess a well-defined long-time state defined by a density matrix $\hat{\rho}_{SS}$ independent of the initial condition $\hat{\rho}(t_{\rm init})$. We use the usual Suzuki-Trotter breakup to express the problem as a two-time contour functional integral [8–10], Hubbard-Stratonovich techniques to decouple the interaction terms by introducing auxiliary fields which we interpret as order parameter fluctuations, and integrate over the electronic degrees of freedom. The result is that the physics of the steady-state system is expressed in terms of a generating functional $\mathcal{Z}$ of source fields η [8,10],

$$\mathcal{Z}(\eta) = \int \mathcal{D}[m_i, m_f]\rho_{SS,\Lambda}(m_i, m_f) \times \int' \mathcal{D}[m_+(t), m_-(t)]e^{S_K[\{m_+, m_-, \eta\}]}. \quad (1)$$

Here $m_\pm$ are the fluctuating order parameter fields of interest and Λ is a short-distance cutoff. $\int' \mathcal{D}[\{m_+(t), m_-(t)\}]$ denotes an integral over all paths in function space beginning at m_i on the + contour at $t = 0$ and ending at m_f at $t = 0$ on the − contour. The contri-

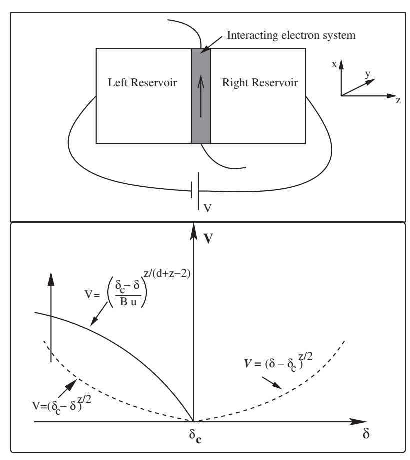

FIG. 1. Upper panel: Schematic view of systems studied; interacting electron system coupled to two leads. Lower panel: Schematic phase diagram in a plane of equilibrium distance from criticality δ and departure from equilibrium V. The quantum critical point δ_c separates the long range ordered ($\delta < \delta_c$) and disordered ($\delta > \delta_c$) phases. The solid line denotes a nonequilibrium phase transition. The dashed curves indicate the crossover from low V essentially equilibrium physics to the higher-V nonequilibrium-dominated regime.

butions of paths with given end points are weighted by the steady-state reduced density matrix $\rho_\Lambda[\{m_i(x), m_f(x)\}]$, whose diagonal elements describe the probability that at an instant of time the long-wavelength components of the order parameter field take the configuration $m(x)$. Equation (1) is the generalization to nonequilibrium systems of the partition function used to treat criticality in equilibrium systems. The important differences are the two-time contours and the presence of ρ_{SS} which obeys a kinetic equation determined [8] from the requirement that correlation functions are causal and finite. Equation (1) may be analyzed by the renormalization group method of integrating out modes near the cutoff and rescaling.

We apply the formalism to the case of a two dimensional itinerant magnet placed between two noninteracting leads, with current flowing across the system (cf. Fig. 1). We model the system by the Hamiltonian

$$H = H_{\text{layer}} + H_{\text{mix}} + H_{\text{leads}}, \tag{2}$$

$$H_{\text{layer}} = \sum_{i,\delta,\sigma} t_\delta c^\dagger_{i+\delta,\sigma} c_{i,\sigma} + H_{\text{int}}, \tag{3}$$

$$H_{\text{mix}} = \sum_{i,k,\sigma,b=L,R} (V^k_b c^\dagger_{i,\sigma} a_{i,k,\sigma,b} + \text{H.c.}), \tag{4}$$

where the combination of the band structure implied by t and the interactions H_{int} is such that the isolated layer is ferromagnetic. The lead electrons are described by operators a and have free fermion correlators $\langle a^\dagger_b a_b\rangle = f_b$ with f_b a Fermi function with lead-dependent chemical potential μ_b. For $V^b_k = 0$ neither the number of electrons nor the magnetization in the layer are conserved.

We assume, as is usual in studies of quantum critical phenomena [1,2], that the electronic propagators and susceptibilities in the layer take the usual Fermi liquid form and treat the interactions by a perturbative renormalization group. The presence of the leads implies that the Green functions describing the propagation of electrons in the interacting layer are

$$G^R(p, \omega) = \frac{1}{\omega - \varepsilon_p - \Sigma^R(p, \omega)} = (G^A)^*, \tag{5}$$

$$G^K(p, \omega) = \frac{\Sigma^K(p, \omega)}{[\omega - \varepsilon_p - \text{Re}\Sigma^R(p, \omega)]^2 + |\text{Im}\Sigma^R(p, \omega)|^2}, \tag{6}$$

where p is a two-dimensional momentum within the layer, $\text{Im}\Sigma^R = \sum_a \Gamma_a$, with $\Gamma_a(p, \omega)$ being the rate at which electrons escape from the active layer into the lead a, and $\Sigma^K = -2i\sum_{a=L,R}\Gamma_a(p, \omega)[1 - 2f(\omega - \mu_a)]$ determines the distribution function imposed by coupling to reservoirs. We shall focus on excitation energies less than Γ_a and momenta less than $\text{Im}\Sigma^R/v_F$ where nonconservation due to escape into the leads is dominant. The analysis sketched above then leads to a generating function of the form of Eq. (1), with $S_K = S^{(2)} + S^{(4)} + \cdots$, where

$$S^{(2)} = -i\int dt \int dt' \int d^d r \int d^d r' (m_{cl}(t, r), m_q(t, r)) \times \begin{pmatrix} 0 & [\chi^{-1}]^A \\ [\chi^{-1}]^R & \Pi^K \end{pmatrix} \begin{pmatrix} m_{cl}(t', r') \\ m_q(t', r') \end{pmatrix}. \tag{7}$$

Here $m_q = \frac{m_- - m_+}{2}$, $m_{cl} = \frac{m_- + m_+}{2}$, the ellipsis denotes terms of higher than fourth order in m, and the fourth order term $S^{(4)}$ will be presented and discussed below. The quadratic-level inverse propagators are

$$[\chi^{-1}]^R(q, \Omega) = ([\chi^{-1}]^A)^* = \delta + \frac{-i\Omega}{\gamma} + \xi_0^2 q^2 + \cdots, \tag{8}$$

$$\Pi^K(q, \Omega) = -2i\sum_{ab} \coth\frac{\Omega + \mu_a - \mu_b}{2T} \frac{(\Omega + \mu_\alpha - \mu_\beta)}{\gamma^{ab}}. \tag{9}$$

The key quantity is Π^K. In equilibrium systems at $T = 0$, $\Pi^K(t)$ vanishes at long times (as a power law for itinerant-electron models); however, at $T \neq 0$ and (for all of the models we have studied) out of equilibrium, $\Pi^K(t \to \infty) \neq 0$. Mathematically, Π_K acts as a mass for the quantum fluctuations. If T or $V = |\mu_a - \mu_b| \neq 0$, quantum fluctuations are gapped and at long times the theory is classical. The $(\chi^{-1})^{(R,A)}$ describe nonconserved (because of the leads) overdamped magnetization fluctuations. $(\gamma_{ab})^{-1} = \langle\Gamma_a(p, \Omega = 0)\Gamma_b(p, \Omega = 0)/\Gamma^3(p, \Omega = 0)\rangle|_{FS}$ are Fermi surface averaged decay rates, and $\gamma^{-1} = \sum_{ab=L,R}\gamma^{-1}_{ab}$. The "mass" (distance from criticality) δ depends on the interaction, layer density of states, and coupling to the leads. The overdamped dynamics implies that even at $V = T = 0$ the momentum conjugate to $m(q, t)$ is logarithmically large, so that the dc fluctuations are essentially classical. The density matrix is then easily obtained from the generating function using the techniques of [10]. We find, up to corrections of $\mathcal{O}(V^2/\Gamma^2)$ in the argument of the exponential,

$$\rho[m_i(k), m_f(k)] \sim \delta_{m_i,m_f} \exp\left[-\frac{2\text{Re}[\chi^{-1}(k)]^R|m_i(k)|^2}{i\gamma\Pi^K(\Omega = 0)}\right]. \tag{10}$$

After calculation, we find that the leading nonlinearity is

$$S^{(4)} = -i\int(d\{k\})\sum_{i=1,\ldots,4} u_i m_q^i m_{cl}^{4-i}. \tag{11}$$

Here the u_i are interaction functions which depend on the momenta and frequency $\{k\}$ of all of the fields. The level broadening due to the leads means any space dependence may be neglected. In an isolated system with Hamiltonian dynamics one would have $u_1 = u_3$ independent of frequency and $u_{2,4} = 0$. The coupling to a reservoir means that the interactions are retarded and that $u_{2,4} \neq 0$. The limit of $u_{1,3}$ as all momenta and frequencies tend to zero is real and positive, so to obtain the leading long-wavelength, low energy behavior we may treat $u_{1,3}$ as constants. However, at $T = V = 0$, $u_{2,4} \to 0$ as the external frequencies are set to zero so the frequency dependence must be considered. This is somewhat involved, but the case important for subsequent considerations is when the two quantum fields carry a frequency $\pm\Omega$; in this case at $V = 0$, $u_2 \to u_2'\Omega\coth\frac{\Omega}{2T}$, while at $T = 0$, $V \neq 0$, $u_2 \to u_2''|V|$, with real and positive iu_2', iu_2''.

We now formulate a renormalization group treatment, following along the usual lines [1,2,4]: we choose a $b > 1$ and in Eq. (7) integrate out those fluctuations with momenta between Λ and Λ/b, treating the interactions perturbatively. One technical remark is needed: causality implies $\langle m_q m_q \rangle$ correlator vanish identically at all times. In a theory with a frequency cutoff, care must be exercised to ensure that the cutoff does not violate the causality requirements. We find it simpler to work with a theory with a momentum cutoff but no frequency cutoff, so that we eliminate all frequencies for each removed momentum mode. We next rescale q, ω, T, V in order to keep the cutoff, coefficients of q^2 and $i\Omega$, and arguments of Π^K invariant; thus $q \to \frac{q'}{b}$, $\Omega \to \frac{\Omega'}{b^z}$, $T, V \to \frac{T', V'}{b^z}$. Observe that under this rescaling the density matrix preserves the form Eq. (10). The result is a change in the mass δ:

$$\frac{d\delta}{d\ln b} = 2\delta + 3u_1(b)g. \tag{12}$$

A similar renormalization coming from u_2 preserves the form of the Π^K term but changes the coefficient. We interpret this as a finite renormalization of the broadening parameter γ and do not consider it further. The mode elimination leads also to a renormalization of the interactions (note: $\bar{u}_{2,4} = iu_{2,4}$, $\epsilon = 4 - d - z$)

$$\frac{du_1}{d\ln b} = \epsilon u_1 - 18u_1^2(f^{KR} + f^{KA}) + 12u_1\bar{u}_2 f^{RA}, \tag{13}$$

$$\frac{d\bar{u}_2}{d\ln b} = \epsilon\bar{u}_2 - 2\bar{u}_2[15u_1(f^{KR} + f^{KA}) - 2\bar{u}_2 f^{RA}] + 18u_1[u_1 f^{KK} - 2u_3 f^{RA}], \tag{14}$$

$$\frac{d(u_1 - u_3)}{d\ln b} = \epsilon(u_1 - u_3) - 12u_1[6u_4 f^{RA} - u_2 f^{KK}] + \cdots, \tag{15}$$

$$\frac{d\bar{u}_4}{d\ln b} = \epsilon\bar{u}_4 - 6\bar{u}_2 u_3(f^{KR} + f^{KA}) + \cdots. \tag{16}$$

Here $g = \frac{K_d}{2}\int\frac{d\omega}{2\pi}\tilde{\chi}_K$; $f^{ab} = \frac{K_d}{4}\int\frac{d\omega}{2\pi}\tilde{\chi}^a\tilde{\chi}^b$, where $\tilde{\chi}^a = \tilde{\chi}^a(\Lambda, \omega)$ is one of χ^R, χ^A, $-i\chi^K = i\Pi^K\chi^R\chi^A$, and $K_d = \int\frac{d^dq}{(2\pi)^d}\delta(q - \Lambda)$ (in our discussions we will set $\Lambda = 1$). Note that the initial values are $u_1 - u_3 = \mathcal{O}(T^2, V^2)$, $\bar{u}_2$, $\bar{u}_4 = \mathcal{O}(T, V)$, and $f^{KK} = 2f^{RA} + \mathcal{O}(T^2, V^2)$. The functions f, g depend on $\delta(T, V)$. When $\delta, T, V \to 0$, f^{ab} and g tend to constant values of order unity. For T, $V > 1$ but $\delta \ll 1$, $f^{KK} \sim V^2, T^2$, $(f^{KR} + f^{KA})$, $g \sim V, T$, and $f^{RA} \sim 1$. When $\delta \sim 1$, scaling stops.

We solve the scaling equations starting from the physically relevant initial conditions δ, T, $V \ll 1$, and taking $d = z = 2$ as appropriate for a thin magnetic layer. To leading nontrivial order in T, V we need to retain only the first terms after the ϵ term in Eqs. (13)–(16). We integrate these equations to obtain $\delta(b) = e^{2\ln b}[\delta_0 + 3g\int_0^{\ln b} dx u_1(e^x)e^{-2x}]$, where $u_1(b) = \frac{u_{10}}{1 + f_1 u_{10}\ln b}$, $f_1 = 18(f^{KR} + f^{KA})$, and the subscript 0 denotes initial conditions. We integrate up to scales where $\delta \sim 1$ (initial conditions corresponding to region below the dashed line in Fig. 1) or, if δ remains small, $\max(V, T) \sim 1$. For the latter case, we denote the value of δ at the crossover scale by r. We focus on the interesting regime $-1 \ll r \leq 0$ corresponding to criticality or to a $T = 0$ ordered state very near to the critical point (scaling trajectory depicted by vertical arrow in Fig. 1). We henceforth set $T = 0$.

At the crossover scale the interactions are all small by powers of logarithms. In the crossover region $V \sim 1$ the expressions are complicated. In the classical region $V \gg 1$, the functions f, g acquire the V dependence noted above. Rewriting the scaling equations in terms of $g = \bar{g}V$, $f^{KK} = \bar{f}^{KK}V^2$, $f^{KR} + f^{KA} = V(\bar{f}^{KR} + \bar{f}^{KA})$, $f^{RA} = \bar{f}^{RA}$, $v_1 = u_1 V$, $v_2 = \bar{u}_2$, $v_3 = \frac{u_3}{V}$, and $v_4 = \frac{\bar{u}_4}{V^2}$ leads to

$$\frac{d\delta}{d\ln b} = 2\delta + 3v_1\bar{g}, \tag{17}$$

$$\frac{dv_1}{d\ln b} = 2v_1 - 18v_1^2(\bar{f}^{KR} + \bar{f}^{KA}) + 12v_1 v_2\bar{f}^{RA}, \tag{18}$$

$$\frac{dv_2}{d\ln b} = 18v_1^2\bar{f}^{KK} - 30v_1 v_2(\bar{f}^{KR} + \bar{f}^{KA}) + 4v_2^2\bar{f}^{RA} - 36v_1 v_3\bar{f}^{RA}, \tag{19}$$

$$\frac{dv_3}{d\ln b} = -2v_3 + \cdots, \qquad \frac{dv_4}{d\ln b} = -4v_4 + \cdots. \tag{20}$$

Thus in the classical regime v_3 and v_4 vanish rapidly, v_1 grows, and v_2 reaches a fixed point. The effective theory becomes quadratic in m_q, which may be integrated out [10]. The result is a theory for the fluctuations of the classical component of the magnetization field in the presence of a Gaussian, delta-correlated noise determined by the nonequilibrium drive:

$$-\frac{1}{\gamma}\frac{\partial m_{cl}}{\partial t} = (\delta - \xi_0^2\nabla^2 + v_1 m_{cl}^2)m_{cl} + \xi,$$
$$\langle\xi(x, t)\xi(x', t')\rangle = \delta(x - x')\delta(t - t')\frac{2T_{\rm eff}}{\gamma}, \tag{21}$$

with $T_{\rm eff} = V\frac{\gamma}{\gamma_{LR}}$ (we have used a standard transformation [11] to eliminate a coupling, generated by v_2, between the noise and the classical field). All parameters acquire finite renormalizations (not explicitly denoted) from the scaling process. Equation (21) is identical to that used in the standard analysis of model A [12] relaxational dynamics, except that $T_{\rm eff}$ appears instead of temperature in the noise correlator. We therefore conclude that the voltage-driven transition is in the same universality class as the usual thermal 2d Ising transition, and more generally that in this model, as far as universal quantities are concerned, voltage acts as a temperature. From Eq. (21) we may extract experimental consequences: near the critical point, magnetic correlation length ξ is $\xi^{-2} = \delta \sim \frac{V}{\ln 1/V}$ and the in-plane 2d resistivity arising from the scattering of electron with critical fluctuations is $\rho(V) \sim V^{3/2}$.

We next extend the analysis to the $O(3)$ symmetric (Heisenberg) case. As in the equilibrium situation [1,2] the physics of the disordered and quantum-classical crossover regimes is only weakly dependent on spin symmetry. Differences appear in the "renormalized classical" regime corresponding to adding a weak nonequilibrium drive to an ordered state. In this regime the procedure leading to Eq. (21) gives a nonlinear stochastic equation describing fluctuations of the magnetization amplitude and precession of its direction. Here we focus on the most important special case, namely, precession of small amplitude, low frequency, long-wavelength fluctuations of the magnetization direction about a state assumed to possess long ranged order directed along $\hat{z}$, and we denote the spin gap by Δ. The important degrees of freedom are those transverse to the ordering direction. We find that at scales $t > 1/\gamma$ and $L > (v_F/\gamma)$ these are described by

$$\left(\frac{a_{xx}}{\gamma} + \hat{z} \times \frac{a_{xy}\Delta}{\gamma^2}\right)\frac{\partial \vec{m}}{\partial t} - \left(b_{xx} - \frac{b_{xy}\Delta V}{\gamma\gamma_{LR}}\hat{z}\times\right)\xi_0^2\nabla^2\vec{m} = \vec{\xi}. \tag{22}$$

$\vec{\xi}$ in Eq. (22) is a fluctuating noise field whose components are independent and correlated according to Eq. (21) but with γ replaced by γ/a_{xx}. The a, b coefficients are numbers of order unity. The subscripts indicate whether they arise from the xx or xy terms in the retarded or advanced susceptibilities. The term involving a_{xy} gives the Landau-Lifshitz-Gilbert spin precession and is explicitly proportional to the spin gap Δ so is small near the quantum critical point. The term involving a_{xx} expresses the damping due to coupling to the leads, remains nonvanishing as $\Delta \to 0$, and is the dominant time derivative term. Solving Eq. (22) in the rotating frame leads to

$$\langle m_+(q,t)m_-(q't')\rangle = \frac{T_{\text{eff}}a_{xx}\delta(q+q')e^{-D_{\text{eff}}q^2\xi_0^2|t-t'|-i\omega_{\text{eff}}(t-t')}}{(a_{xx}b_{xx} - a_{xy}b_{xy}\frac{\Delta^2 V}{\gamma^2\gamma_{LR}})q^2\xi_0^2}. \tag{23}$$

Equation (23) shows that the mean square magnetization fluctuations diverge as $\frac{1}{q^2}$. This divergence signals the instability of the ordered state by the voltage-induced decoherence in precise analogy to the usual 2d thermal case. The oscillatory term in Eq. (23) expresses the usual spin precession with precession frequency $\omega_{\text{eff}} = (\frac{a_{xy}b_{xx}\gamma^2 + a_{xx}b_{xy}V(\gamma^2/\gamma_{LR})}{a_{xx}^2\gamma^2 + a_{xy}^2\Delta^2})\Delta q^2\xi_0^2$ shifted from the equilibrium result by an amount proportional to V due to spin accumulation effects at the interface of the magnetized layer and leads [13]. The decaying term in Eq. (23) expresses the damping due to coupling to leads $D_{\text{eff}} = \frac{[a_{xx}b_{xx}\gamma^3 - a_{xy}b_{xy}\Delta^2 V(\gamma/\gamma_{LR})]}{a_{xx}^2\gamma^2 + a_{xy}^2\Delta^2}$. If $a_{xx}b_{xx} < a_{xy}b_{xy}\frac{\Delta^2 V}{\gamma_{LR}\gamma^2}$, Eq. (22) supports modes which grow exponentially with time leading to the spin-torque instability recently discussed [14]. However, where the present theory applies ($\frac{\Delta,V}{\gamma} \ll 1$) there is no instability. Calculation reveals that obtaining a nonzero b_{xy} also requires an energy-dependent asymmetry between the leads $[\Gamma_L(\varepsilon_1)\Gamma_R(\varepsilon_2) - \Gamma_L(\varepsilon_2)\Gamma_R(\varepsilon_1) \neq 0]$.

We have presented a theory for nonequilibrium phase transitions in an itinerant-electron system coupled to external reservoirs. We provide a precise mapping onto an effective classical theory which demonstrates that the leading effect of the nonequilibrium drive is to generate an effective temperature and hence a transition in the standard Wilson-Fisher thermal universality class. Nonequilibrium-induced breaking of time reversal and inversion symmetries and the creation of a coherently precessing ("spin-torque") state appear only at the level of subleading corrections. The techniques introduced here can be applied to important open problems such as systems where the drive couples linearly to the order parameter [5,6], driven Bose condensates [15], and the closed system, conserved order parameter work of [7].

This work was supported by NSF-DMR-0431350 (A. J. M.) and NSERC, CRC, CIAR, KRF-2005-070-C00044 (A. M., S. T., Y. B. K.).

Note added.—Shortly after our manuscript was submitted a study of the closely related problem of nonlinear transport at a quantum critical point appeared [16].

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